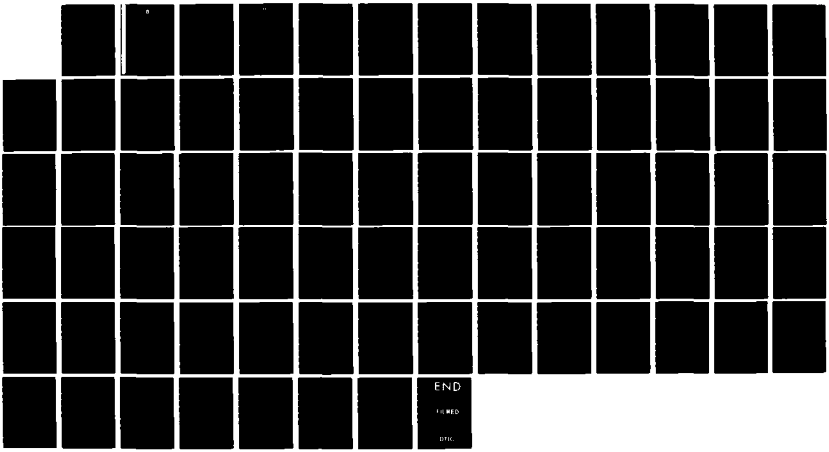
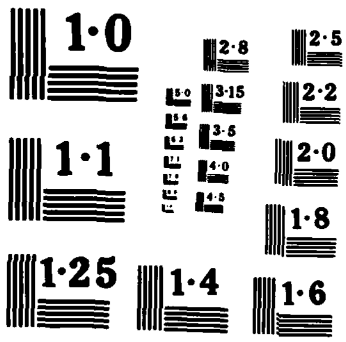
AD-A158 860 EVOLUTION OF NONLINEAR WAVE GROUPS ON WATER OF SLOWLY-VARYING DEPTH(U) TECHNION - ISRAEL INST OF TECH HAIFA DEPT OF CIVIL ENGINEERING M STIASSNIE ET AL. JUN 85 DAJA37-82-C-0300 1/1

UNCLASSIFIED F/G 20/4 NL





NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

EVOLUTION OF NONLINEAR WAVE GROUPS ON WATER OF SLOWLY-VARYING DEPTH

Michael Stiassnie & Ruth Iusim

Department of Civil Engineering

Technion - Israel Institute of Technology

Contract No. DAJA 37-82-C-0300

Final Report - June 1985

The Research reported in this document has been made possible through the support and sponsorship of the U.S. Government through its European Research Office of the U.S. Army.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Evolution of nonlinear wave groups on water of slowly-varying depth.		5. TYPE OF REPORT & PERIOD COVERED Final Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Michael Stiassnie and Ruth Lusim		8. CONTRACT OR GRANT NUMBER(s) DAJA37-82-C-0300
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Civil Engineering Technion - Israel Institute of Technology		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102A ILI6LL02BH57 01
11. CONTROLLING OFFICE NAME AND ADDRESS USARDSG-UK PO Box 65 FPO NY, NY 09510		12. REPORT DATE June, 1985
		13. NUMBER OF PAGES 71 (seventy one)
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Water waves; Wave groups; Hydraulics; Shoaling water; Non-linear waves.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Development of Mathematical formulation which describes the evolution of non-linear wave groups in water of slowly varying depths. Report summarizes the derivations of the appropriate equations and their solutions. An approximate analytical solution is provided giving the physical processes involved in the shoaling of wave groups. Also, calculations of the induced mean flow accompanying an envelope soliton moving on on deepwater.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

EVOLUTION OF NONLINEAR WAVE GROUPS ON WATER OF SLOWLY-VARYING DEPTH

Michael Stiassnie & Ruth Iusim

Department of Civil Engineering

Technion - Israel Institute of Technology

Contract No. DAJA 37-B2-C-0300

Final Report - June 1985

The Research reported in this document has been made possible through the support and sponsorship of the U.S. Government through its European Research Office of the U.S. Army.

TABLE OF CONTENTS

	Page
Nomenclature	1
1. Introduction and Achievements	6
2. Derivation of the Evolution Equations	8
(2.1) Fourth order evolution equations	8
(2.2) Third order evolution equations	18
(2.3) Third order equations obtained from Whitham equations	21
(2.3a) Modulation equations	21
(2.3b) Induced mean flow	22
(2.3c) Nonlinear Schrödinger equation	25
3. Solutions	27
(3.1) Shoaling of wave groups	28
(3.1a) Three waves systems	31
(3.1b) Numerical verification of the asymptotic solution	33
(3.1c) On P and I_o	35
(3.1d) Group envelopes	36
(3.1e) The mean flow field	37
(3.2) The induced flow accompanying an envelope soliton	39
4. Concluding remarks	42
Appendices:	
A - Decoupling of eqs. (3.3), (3.4) for periodical boundary conditions	43
B - Solutions of eqs. (3.1a.3),(3.1a.4) for constant depth	47
C - Derivation of the equation for $J_3(X)$	48
D - Solution of the Neumann Problem (3.2.6), (3.2.7) (3.2.11)	54
References	58
List of Figures	59
Figures	60

NOMENCLATURE

a - wave amplitude

a_n - see eq. (3.1.2)

$A(\xi,\tau)$ - see eq. (2.1.8)

$A_1(\xi,\tau)$ - see eq. (2.1.52)

b - see eq. (B.3d)

b_n - see eq. (3.1.3)

C_p - phase velocity - see eq. (2.1.36)

Cg - group velocity - see eq. (2.1.28)

C - see eq. (2.3a.5)

c_n - see eq. (3.1.4)

$\tilde{c}_1,\tilde{c}_2$ - see eqs. (B.1), (B.2)

c - see eq. (B3.a)

cn - Jacobian Ellyptic Function - see eqs. (C.15)

cd - Jacobian Ellyptic Function - see efter eq. (C.12)

d - see eq. (B.3b)

dn - Jacobian Ellyptic Function - see eqs. (C.15)

$D(\xi,\tau)$ - see eq. (2.1.19)

$Dn(X)$ - see eq. (3.1a.1)

e - see eq. (B3.c)

$E(\xi,\tau)$ - see eq. (2.1.23)

$E, E(p,r)$- Ellyptic Integrals - see Byrd and Friedman (1971)

$F(\xi,\tau)$ - see eq. (2.1.22)

$F(p,r)$ - Ellyptic Integral - see Byrd and Friedman(1971)

$f(\lambda,z)$ - see eq. (D.1)

g -	gravity constant
$G(\xi,\tau)$ -	see eq. (2.1.20)
G_1,G_2 -	see eq. (2.3b.4)
h -	still water depth
$\tilde{h}$ -	water depth
i -	complex unity
$I(P)$ -	see eq. (3.1a.6)
J_1,J_2,J_3 -	see eqs. (3.1.12),(3.1.13),(3.1.14)
k -	carrier wave number - see eq. (2.1.8)
$\tilde{k}$ -	wave number - see eq. (2.3a)
$K(P)$ -	see after eq. (C.18)
ℓ_1 -	wave length, see eq. (3.1d.1)
ℓ_2 -	group length, see eq. (3.1d.2)
ℓ_3 -	'supergroup length' - see eq. (3.1d.3)
$M(\xi,\tau)$ -	see eq. (3.1.21)
P -	see eq. (3.1.17)
p -	see eq. (C.19)
q -	see eq. (C.17)
$Q(\xi)$ -	see eq. (2.2.5)
$Q_1(\xi)$ -	see eq. (A.2)
$Q_2(\xi)$ -	see eq. (2.2.9)
r -	see eq. (B.4)
r' -	see eqs. (C.15)
sn -	Jacobian Ellyptic Function - see eqs (C.14)
t -	time

T -	dimensionless form of τ.
	in section (3.1) - see (3.1.17)
	in section (3.2) - see (3.2.5)
U -	horizontal velocity
u -	dimensionless form of U - see (3.2.14)
V -	vertical velocity
v -	dimensionless form of v (see (3.2.15))
w(x) -	see eq. (D.9)
x -	horizontal coordinate
X -	dimensionless for of ξ
	in section (3.1) - see eq. (3.1.I7)
	in section (3.2) - see eq. (3.2.5)
y -	see eq. (B.5)
z -	vertical coordinate
$\tilde{z}$ -	see eq. (B.6)
Z -	dimensionless form of z
	in section (3.1) - see after eq. (3.1e.1)
	in section (3.2) - see eqs. (3.2.5)
$Z_J(p,r)$ -	Jacobian Z function - see eq. (C.25)
α -	see eq. (3.1.10)
α_1,α_2 -	see eqs. (2.2.13),(2.2.14)
α_n -	see eq. (3.2.13)
β -	see eq. (3.1.10)
β_1,β_2,β_3 -	see eqs. (3.5),(3.6),(3.7)

γ - see after eq. (3.1.1)

$\Gamma_1, \Gamma_2 \ldots$ - see eqs. (A.8), (A.9)

δ - see eq. (C.24)

ε - see after eq. (2.1.4)

ζ - free water surface

$\zeta_o, \zeta_1, \zeta_2 \ldots$ see eq. (2.1.7)

$\zeta_{mj}(\xi,\tau)$, $m \geq 1$, $j \geq m$ - see eq. (2.1.10)

ζ_{20} - see eq. (2.1.17)

θ - see eq. (2.1.8)

λ_j, $j=1,9$ - see eqs. (2.1.43) to (2.1.51)

$\mu(\xi)$ - see eq. (3.1.9)

μ_j, $j=1,6$ - see eqs. (2.1.43) to (2.1.51)

ξ - see eq. (2.1.15)

σ - see eq. (2.1.26)

$\tilde{\sigma}$ - see after eq. (2.3a.4)

τ - see eq. (2.1.15)

φ - see eq. (2.3c.6)

$\tilde{\varphi}$ - see eq. (C.30)

ϕ - induced velocity potential

$\phi_o, \phi_1, \phi_2 \ldots$ - see eq. (2.1.16)

$\phi_{mj}(\xi,\tau,z)$, $m \geq 1$, $j \geq m$ - see eq. (2.1.9)

ϕ_{10}, ϕ_{20} - see eq. (2.1.11)

$\tilde{\phi}$ - dimensionless form of ϕ_{10} - see eq. (3.2.5)

$\tilde{\psi}$ - wave envelope

ψ -	dimensionless form of $\tilde{\psi}$
	in section (3.1) see eq. (3.1.6)
	in section (3.2) see eq. (3.2.5)
Ψ -	stream function
$\tilde{\Psi}$ -	dimensionless form of Ψ
	in section (3.1) see after eq. (3.1e.1)
	in section (3.2) see eq. (3.2.16)
ω -	carrier wave frequency - see eq. (2.1.8)
$\tilde{\omega}$ -	wave frequency - see after eq. (2.3a.4)

1. INTRODUCTION AND ACHIEVEMENTS

The shoaling of weakly nonlinear surface wave groups is important to the understanding of coastal wave climate and coastal flow regime.

In the past, most efforts concentrated on the equally important though simpler problem of shoaling of wave-trains (i.e. monochromatic wave groups), for details see Stiassnie & Peregrine (1980).

The first mathematical formulation for shoaling of wave-groups was given by Djordjevic' and Redekopp (1978). This formulation is limited to cases where the water depth is small compared to the group-length. Equations suitable for water depths of the order of the group-length are deduced in Peregrine (1983); combining the constant depth model by Davey and Stewartson (1974) and the higher-order model for infinitely deep water by Dysthe (1979).

The only available solutions are those for the shoaling of isolated wave-packets (solitons), which were originally given by Djordjevic' and Redekopp in their 1978 paper. They predict that a soliton envelope can undergo fission only if it propagates into deeper water. By heuristic assumptions for the evolution along the slope, they also estimate the number of solitons emitted after a single soliton descends from a shallower shelf. A more recent study, Turpin, Benmoussa and Mei (1983) confirms these results qualitatively, but not quantitatively.

To the best of our knowledge, no results for shoaling of wave groups (i.e. modulated wave-trains) have been presented so far. These modulated wave-trains are of particular importance since almost every wave-train will eventually become modulated due to its intrinsic Benjamin-Feir instability. The main goal of the present study is to throw light on the evolution during the shoaling of a modulated wave-train and its influences on the mean free surface and the wave-induced mean flow.

The bulk of this work is divided into two parts: the derivation of the appropriate equations (in chapter 2), and their solutions (in chapter 3).

The use of the REDUCE 2 algebraic manipulator enabled us to derive evolution equations which are accurate to <u>fourth order</u>. These equations are valid for any water depth, (except the extremely shallow), as long as the slopes of the bottom are sufficiently mild. Note the nonlinear effects on shoaling surface gravity waves in extremely shallow water are discussed in a recent paper by Freilich and Guza (1984).

Two new results are presented and discussed in this report. The first is an approximate analytical solution which provides detailed information about the physical processes involved in the shoaling of wave-groups. The second result is the calculation of the induced mean flow accompanying an evelope-soliton moving on infinitely deep water.

The attempts to include the effects of randomness into our models have not yielded any wortwhile results. It seems that the very recent work by Longuet-Higgins (1984) should provide a good starting point for any future work in this direction.

2. DERIVATION OF THE EVOLUTION EQUATIONS

2.1 Fourth Order Evolution Equations

The third order evolution equations for groups of water-waves moving over an uneven bottom were first derived by Djordjevic' and Redekopp (1978). These equations are valid only if the water depth h is of the same order of magnitude as the wave length ℓ_1. In the present section we carry their derivation one step further, to fourth order and introducing a slight modification, we obtain a more general set of equations which is valid when $O(h/\ell_1) > 1$. All the algebraic manipulations were performed utilizing the REDUCE 2 algebraic manipulator, see Hearn (1973).

We consider the evolution of a uni-directional progressive gratitvy wave moving along the x axis on the free surface of a homogeneous liquid with depth $h = h(x)$ varying in the direction of the propagation. The effect of surface tension is neglected, so the analysis applies to gravity waves only. The fluid motion is irrotational, thus there exists a velocity potential $\phi(x,z,t)$ satisfying Laplace's equation

$$\phi_{xx} + \phi_{zz} = 0 \qquad (2.1.1)$$

where z is the vertical coordinate, and $z = 0$ is the undisturbed free surface.

The boundary condition on the bottom $z = -h(x)$ is

$$\phi_z = -h'(x)\ \phi_x \qquad (2.1.2)$$

and the boundary condition on the free surface $z = \zeta(x,t)$ are the kinematic condition:

$$\phi_z = \zeta_t + \phi_x \zeta_x \tag{2.1.3}$$

and the pressure condition

$$2g\zeta + 2\phi_t + \phi_x^2 + \phi_z^2 = 0 \tag{2.1.4}$$

The situation where the depth varies slowly in the direction of propagation is considered, so that properties characterizing the wave will change slowly as well. A small nondimensional parameter ε that measures the slope of the wavy surface is introduced, and we define the new variables

$$\tau = \varepsilon\left[\int^{x} \frac{dx}{C_g(\xi)} - t\right] \; ; \; \xi = \varepsilon^2 x \tag{2.1.5}$$

where C_g is the group velocity.

We suppose that the depth changes on the scale of ε^2 so that $h = h(\xi)$ with the property $h'(\xi) = O(1)$.

The velocity potential and the free surface displacement are expanded as follows:

$$\phi = \phi_o(\xi,\tau,z)+\{\phi_1(\xi,\tau,z)e^{i\theta}+\phi_2(\xi,\tau,z)e^{2i\theta}+\ldots+c.c\} \tag{2.1.6}$$

$$\zeta = \zeta_o(\xi,\tau)+\{\zeta_1(\xi,\tau)e^{i\theta}+\zeta_2(\xi,\tau)e^{2i\theta}+\ldots+c.c\} \tag{2.1.7}$$

where $\theta = \int^{x} k(\xi)dx - \omega t$ (2.1.8)

and c.c means complex conjugate.

With ε chosen to be small, the functions $\phi_j(\xi,\tau,z)$ and $\zeta_j(\xi,\tau)$ for $j > 1$ are expanded formally in powers of ε as follows:

$$\phi_j(\xi,\tau,z) = \sum_{m=j}^{\infty} \varepsilon^m \phi_{mj}(\xi,\tau,z) \tag{2.1.9}$$

$$\zeta_j(\xi,\tau) = \sum_{m=j}^{\infty} \varepsilon^m \zeta_{mj}(\xi,\tau) \tag{2.1.10}$$

The induced mean flow potential ϕ_o is written as

$$\phi_o(x,z,t) = \varepsilon\ \phi_{10}(\xi,\tau,z) + \varepsilon^2\ \phi_{20}(\xi)\cdot t \tag{2.1.11}$$

where $0(\phi_{10}) \leqslant 1$.

Substituting the Fourier series (2.1.6) into the Laplace equation (2.1.1) we obtain that the zero order potential satisfies:

$$\underline{\underline{\nabla^2 \phi_o(\xi,\tau,z) = 0}} \tag{2.1.12}$$

From (2.1.11) and (2.1.12) it follows that:

$$\phi_{o_{zz}}(\xi,\tau,z) \leqslant 0(\varepsilon^3) \tag{2.1.13}$$

Expanding the free surface conditions (2.1.3) and (2.1.4) around the equilibrium position $z = 0$, substituting eqs. (2.1.6) to (2.1.10) and looking for the coefficient of e^o, yields respectively:

$$\phi_{10_z} + \zeta_{o_\tau} + 2\varepsilon^2\cdot \mathrm{Re}[\zeta_{11}\phi^*_{21_{zz}} + \zeta_{21}\phi^*_{11_{zz}} + \frac{ki}{C_g}\zeta_{11}\phi^*_{11_\tau} -$$

$$- k^2\phi_{11}\zeta^*_{21} - \frac{ik}{C_g}\zeta_{11}\phi^*_{11_\tau} - k^2\zeta_{11}\phi^*_{21}] + 0(\varepsilon^4) = 0 \tag{2.1.14}$$

and

$$\zeta_o = \frac{\varepsilon^2}{g}\phi_{10_\tau} + 2\mathrm{Re}\frac{\varepsilon^2}{g}[i\omega\phi_{11_z}\zeta^*_{11}] - \frac{\varepsilon^2}{g}[k^2\phi_{11}\phi^*_{11} + \phi_{11_z}\phi^*_{11_z}] + 0(\varepsilon^3) \tag{2.1.15}$$

where $\phi_{10} = \phi_{10}(\xi,\tau,o)$, $\phi_{ij} = \phi_{ij}(\xi,\tau,o)$, and * denotes the complex conjugate.

From (2.1.14) and (2.1.15) it follows that the order of ζ_o and $\phi_{10_z}(o)$ are greater or equal to ε^2

$$\phi_{10_z}(o) \leqslant 0(\varepsilon^2); \qquad \zeta_o \leqslant 0(\varepsilon^2) \tag{2.1.16a,b}$$

We introduce the notation

$$\zeta_o = \varepsilon^2 \zeta_{20} \tag{2.1.17}$$

The next step of the derivation is similar to that given by Djordjevic' and Redekopp with the difference that we continue the process to fourth order. Substitution of (2.1.6), (2.1.8) and (2.1.9) into Laplace's equation (2.1.1) gives:

$\underline{\underline{0(\varepsilon)e^{i\theta}}}$:

$$\phi_{11} = A(\xi,\tau)\,\frac{\cosh k(z+h)}{\cosh k\,h} \tag{2.1.18}$$

$\underline{\underline{0(\varepsilon^2)e^{i\theta}}}$:

$$\phi_{21} = D(\xi,\tau)\,\frac{\cosh k(z+h)}{\cosh k\,h} - \frac{iA_\tau}{C_g}\,\cdot$$

$$\cdot\,\frac{(z+h)\,\sinh k(z+h) - h\sigma\,\cosh k(z+h)}{\cosh k\,h} \tag{2.1.19}$$

$\underline{\underline{0(\varepsilon^3)e^{i\theta}}}$:

$$\phi_{31} = G(\xi,\tau)\,\frac{\cosh k(z+h)}{\cosh k\,h} -$$

$$-\frac{i}{2\cosh k\,h}\{2k\,h'A+(k'A - \frac{i}{C_g^2}A_{\tau\tau})(z+h)\}\cdot$$

$$\cdot\,(z+h)\cosh k(z+h) + \frac{i}{\cosh k\,h}\{\sigma(hk)'A -$$

$$-A_\xi - \frac{ih\sigma}{C_g^2}\,A_{\tau\tau} - \frac{1}{C_g}\}D_\tau\cdot(z+h)\sinh k(z+h) \tag{2.1.20}$$

$\underline{O(\varepsilon^4)e^{i\theta}}$:

$$\phi_{41} = M(\xi,\tau)\,\frac{\cosh k(z+h)}{\cosh k\,h} +$$

$$+\left[\frac{i}{6C_g^3}A_{\tau\tau\tau} - \frac{k'}{2C_g}A_\tau\right]\left[(z+h)^3\right]\frac{\sinh k(z+h)}{\cosh k\,h} -$$

$$-\frac{kh'}{C_g}A_\tau(z+h)^2\,\frac{\sinh k(z+h)}{\cosh k\,h} +$$

$$+\Big\{\frac{1}{2kC_g}\,\{(kh')\left[3\sigma-2kh(2\sigma^2-1)\right] + kh'\sigma - hk'\sigma -$$

$$-2kh\sigma\cdot\frac{C_g'}{C_g} + \frac{2\sigma^2}{1-\sigma^2}\frac{k'}{k}\}\,A_\tau + \frac{h\sigma}{C_g}A_{\tau\xi} -$$

$$-\frac{i\sigma^2}{1-\sigma^2}\frac{k'}{k}D - iD_\xi - \frac{iG_\tau}{C_g}\Big\}\cdot\,(z+h)\,\frac{\sinh k(z+h)}{\cosh k\,h}$$

$$+\Big\{\frac{1}{4kC_g}\left[\frac{-4\sigma^2}{1-\sigma^2}k' + 2kh\sigma k' + 2k\frac{C_g'}{C_g}\right]A_\tau -$$

$$-\frac{ih\sigma}{2C_g^3}A_{\tau\tau\tau} - \frac{A_{\xi\tau}}{C_g}\frac{D_{\tau\tau}}{2C_g^2} - \frac{ik'}{2}D\Big\}\,(z+h)^2\,\frac{\cosh k(z+h)}{\cosh k\,h} +$$

$$+\Big\{-\frac{h'}{C_g}(1-kh\sigma)A_\tau - ikh'D\Big\}\,(z+h)\,\frac{\cosh k(z+h)}{\cosh k\,h} \qquad (2.1.21)$$

$\underline{O(\varepsilon^2)e^{2i\theta}}$:

$$\phi_{22} = F(\xi,\tau)\,\frac{\cosh 2k(z+h)}{\cosh 2\,k\,h} \qquad (2.1.22)$$

$\underline{O(\varepsilon^3)e^{2i\theta}}$:

$$\phi_{32} = E(\xi,\tau)\,\frac{\cosh 2k(z+h)}{\cosh 2\,k\,h} - \frac{iF_\tau}{C_g}\,\cdot$$

$$\cdot\,(z+h)\,\sinh\frac{2k(z+h)}{\cosh 2\,k\,h} \qquad (2.1.23)$$

where A,D,G,F,E, and M are yet unknown functions.

Substitution of eq. (2.1.6) to (2.1.10) and eqs. (2.1.18) to (2.1.23) into the expansion around z = 0 of the free surface conditions (2.1.3) and (2.1.4) gives:

$\underline{O(\varepsilon)e^{i\theta}}$:

$$\zeta_{11} = \frac{i\omega}{g} A \tag{2.1.24}$$

and

$$\omega^2 = g\ k\ \sigma \tag{2.1.25}$$

where

$$\sigma = \mathrm{tgh}(kh) \tag{2.1.26}$$

$\underline{O(\varepsilon^2)e^{o}}$:

$$\zeta_{20} = \frac{1}{g}(\phi_{10\tau} - \phi_{20}) - \frac{k^2}{g}|A|^2 \tag{2.1.27}$$

$\underline{O(\varepsilon^2)e^{i\theta}}$:

$$C_g = \frac{g}{2\omega}[\sigma + kh(1-\sigma^2)] \tag{2.1.28}$$

and

$$g\zeta_{21} = i\omega D + A_\tau \tag{2.1.29}$$

$\underline{O(\varepsilon^2)e^{2i\theta}}$:

$$g\zeta_{22} = -\frac{k^2}{2}\ \frac{(3-\sigma^2)}{\sigma^2}\ A^2 \tag{2.1.30}$$

and

$$\omega\ F = \frac{3}{4}\ ik^2\ \frac{(1-\sigma^4)}{\sigma^2}\ A^2 \tag{2.1.31}$$

$\underline{O(\varepsilon^3)e^{i\theta}}$:

$$\mu_1 A + \mu_2 A_\xi + \mu_3 A_{\tau\tau} + \mu_4 A\phi_{10_\tau} +$$

$$+ \mu_5 |A|^2 A + \mu_6 \phi_{20} A = 0 \tag{2.1.32}$$

where

$$\mu_1 = -i\{\frac{\sigma^2}{1-\sigma^2}\frac{k'}{k}[\sigma + hk(1-\sigma^2)] + (hk)'\} \tag{2.1.33}$$

$$\mu_2 = -\frac{2\omega i C_g}{g} \tag{2.1.34}$$

$$\mu_3 = \frac{1}{g}\left[1 - \frac{gh}{C_g^2}(1-kh\sigma)(1-\sigma^2)\right] \tag{2.1.35}$$

$$\mu_4 = \frac{k^2}{g}\left[2\frac{C_p}{C_g} + (1-\sigma^2)\right]\ ; \quad C_p = \omega/k \tag{2.1.36}$$

$$\mu_5 = \frac{k^4}{2g}\left[\frac{9}{\sigma^2} - 12 + 13\sigma^2 - 2\sigma^4\right] \tag{2.1.37}$$

$$\mu_6 = \frac{2\omega k^2}{g^2}(1-\sigma^2) \tag{2.1.38}$$

and

$$\begin{aligned} g\zeta_{31} &= i\omega G + \left(1+\frac{\omega h\sigma}{C_g}\right)D_\tau + \left(\frac{i\omega k\sigma}{g} - \frac{ik}{C_g}\right)\phi_{o_\tau}A + \\ &+ \frac{i\omega h^2}{2C_g^2}(2\sigma^2-1)A_{\tau\tau} + h\omega\sigma A_\xi - \\ &- \frac{ik^4}{2\omega\sigma^2}\{3-9\sigma^2+11\sigma^4-2\sigma^6\}A^2\bar{A} + \\ &+ \frac{h\omega}{2}\left[2kh' + k'h - 2\sigma^2(kh)'\right]A \end{aligned} \tag{2.1.39}$$

$\underline{O(\sigma^3)e^{2i\theta}}$:

$$\begin{aligned} \frac{4\sigma^3}{1+\sigma^2}E &= \frac{k}{2\omega C_g}\left[3kh(3\sigma^4+2\sigma^2-5)+9\sigma(1-\sigma^2)\right]A\,A_\tau \\ &+ \frac{6ik\omega}{g}(1-\sigma^2)AD \end{aligned} \tag{2.1.40}$$

and

$$\begin{aligned} g\zeta_{32} &= \frac{ik}{4C_g\sigma^3}\left[-12kh(1-\sigma^2)-4\sigma^3+12\sigma\right]A\,A_\tau \\ &- \frac{k^2}{\sigma^2}(3-\sigma^2)AD \end{aligned} \tag{2.1.41}$$

$\underline{\underline{O(\varepsilon^4)e^{i\theta}}}$:
$$\mu_1 D + \mu_2 D_\xi + \mu_3 D_{\tau\tau} + \mu_4 D\phi_{10_\tau} + 2\mu_5 |A|^2 D + \mu_5 A^2 \bar{D}$$
$$+ \mu_6 \phi_{20} D + \lambda_1 A_\tau + \lambda_2 A_{\tau\tau\tau} + \lambda_3 |A|^2 A_\tau + \lambda_4 A^2 \bar{A}_\tau +$$
$$+ \lambda_5 A_{\xi\tau} + \lambda_6 A\phi_{10_{\tau\tau}} + \lambda_7 A_\tau \phi_{10_\tau} + \lambda_8 A\phi_{10_\xi} +$$
$$+ \bar{\varepsilon}^2 \lambda_9 A\phi_{10_z} = 0 \qquad (2.1.42)$$

where $\mu_1,\ldots,\mu_6$ are given by (2.1.32) to (2.1.33) and

$$\lambda_1 = -\frac{h' g^2}{4\omega^2 C_g^3} \left(2h^3k^3\sigma^7 - 6h^3k^3\sigma^5 + 6h^3k^3\sigma^3 - 2h^3k^3\sigma - \right.$$
$$- 5h^2k^2\sigma^6 + 9h^2k^2\sigma^4 - 5h^2k^2\sigma^2 + h^2k^2 + 4hk\sigma^5 -$$
$$\left. -4hk\sigma^3 - \sigma^4 + \sigma^2\right) \qquad (2.1.43)$$

$$\lambda_2 = \frac{ih^2(1-\sigma^2)}{3C_g^3}\left[-3\sigma + kh(3\sigma^2-1)\right] \qquad (2.1.44)$$

$$\lambda_3 = \frac{ik^3}{2gC_g\sigma^3}\{7\sigma^7 - 47\sigma^5 + 48\sigma^3 - 36\sigma + kh\,[-7\sigma^8 + 32\sigma^6 -$$
$$-25\sigma^4 - 18\sigma^2 + 18] \qquad (2.1.45)$$

$$\lambda_4 = \frac{ik^3}{2gC_g}\{-\sigma^4 + 5\sigma^2 + kh\sigma(1-\sigma^2)^2\} \qquad (2.1.46)$$

$$\lambda_5 = \frac{-2h(1-\sigma^2)}{C_g}(1-kh\sigma) \qquad (2.1.47)$$

$$\lambda_6 = \frac{i\omega k\sigma}{g^2} - \frac{2ik}{gC_g} - \frac{i\omega}{gC_g^2} \qquad (2.1.48)$$

$$\lambda_7 = -\frac{ik}{gC_g}\{\frac{2C_p}{C_g} + 2[1-kh\sigma][1-\sigma^2] + 2\} \tag{2.1.49}$$

$$\lambda_8 = \frac{2\omega}{g} k \tag{2.1.50}$$

$$\lambda_9 = -\frac{i\sigma k\omega}{g} \tag{2.1.51}$$

Combining eqs. (2.1.31) and (2.1.40) and introducing the new variable

$$A_1 = \varepsilon A + \varepsilon^2 D \tag{2.1.52}$$

one finally obtains the modification to fourth order of the cubic Schrödinger equation

$$\mu_1 A_1 + \mu_2 A_{1\xi} + \mu_3 A_{1_{\tau\tau}} + \mu_4 A_1 \phi_{10_\tau} + \varepsilon^{-2}\mu_5 |A_1|^2 A_1 +$$

$$+ \mu_6 A_1 \phi_{20}(\xi) + \varepsilon\lambda_1 A_{1_\tau} + \varepsilon\lambda_2 A_{1_{\tau\tau\tau}} + \varepsilon^{-1}\lambda_3 |A_1|^2 A_{1_\tau} +$$

$$+ \varepsilon^{-1}\lambda_4 A_1^2 \bar{A}_{1_\tau} + \varepsilon\lambda_5 A_{1_{\xi\tau}} + \lambda_6 \varepsilon A_1 \phi_{10_{\tau\tau}} + \varepsilon\lambda_7 A_{1_\tau}\phi_{10_\tau} +$$

$$+ \lambda_8 \varepsilon A_1 \phi_{10_\xi} + \varepsilon^{-1}\lambda_9 A_1 \phi_{10_z} = 0 \tag{2.1.53}$$

On the other hand, substitution of (2.1.18), (2.1.24) and (2.1.11) into (2.1.14) and (2.1.15) and elimination of ζ_o, yields the following equation for ϕ_{10}:

$$\phi_{10_z} + \frac{\varepsilon^2}{g}\phi_{10\tau\tau} = \frac{\varepsilon^2 k^2}{g}[\frac{2C_p}{C_g} + (1-\sigma^2)]\{|A|^2 + \varepsilon AD^* + \varepsilon A^* D\}_\tau$$

$$+ \frac{2\omega}{g}\varepsilon^3 \frac{\partial}{\partial\xi}[k|A|^2] - \varepsilon^3[-\frac{i\omega}{gC_g^2} + \frac{2i\omega k\sigma}{g^2} - \frac{2ik}{gC_g}][AA^*_{\tau\tau} - A^*A_{\tau\tau}] + 0(\varepsilon^4)$$

$$\tag{2.1.54}$$

which can be written to the same order of accuracy as:

$$\phi_{10_z} + \frac{\varepsilon^2 \phi_{10_{\tau\tau}}}{g} = \frac{k^2}{g}[\frac{2C_p}{C_g} + (1-\sigma^2)](|A_1|^2)_\tau + \frac{2\omega}{g}\varepsilon \cdot$$

$$\cdot \frac{\partial}{\partial\xi}[k|A_1|^2] - \varepsilon[-\frac{i\omega}{gC_g^2} + \frac{2i\omega k\sigma}{g^2} - \frac{2ik}{gC_g}][A_1 A^*_{1_{\tau\tau}} - A_1^* A_{1_{\tau\tau}}] \tag{2.1.55}$$

Laplace's equation (2.1.12) and the boundary conditions (2.1.2) at $z = -h$ and eqs. (2.1.51), (2.1.53) at $z = 0$ form the system of equations from which A_1 and Φ_{10} can be determined.

For the case of infinitely deep water, as it will be shown later, the induced flow ϕ_o is of order ε^2. For this case equations (2.1.55) and (2.1.53) become respectively:

$$\phi_{10_z} = \frac{4k^2}{g} \left(|A_1|^2\right)_\tau \tag{2.1.56}$$

and

$$- \frac{2\omega i C_g}{g} A_{1_\xi} + \frac{A_{1_{\tau\tau}}}{g} + \varepsilon^{-1} \frac{2\omega k}{g C_g} A_1 \phi_{10_\tau} +$$

$$+ \varepsilon^{-1} \frac{4k^4}{g} |A_1|^2 A_1 + \varepsilon^{-1} \Big[- \frac{14 i k^3}{g C_g} |A_1|^2 A_{1_\tau} +$$

$$+ \frac{2ik^3}{gC_g} A_1^2 \bar{A}_{1_\tau} \Big] - \varepsilon^{-1} \frac{i\omega k}{g} A_1 \phi_{10_z} = 0 \tag{2.1.57}$$

The last two equations are identical to those given in K.B. Dysthe (1980).

<u>Note</u> that the following typographical errors were found in Dysthe's paper:

His equations (2.17) and (2.19) should be written (in his notation) as follows:

$$\Gamma = 4k^4 |A|^2 + 8ik^3 (AA_x^* - A^* A_x) - 4ik^3 |A|_x^2 + 2\omega k (\bar{\phi}_x - i\bar{\phi}_z)$$

(Dysthe 2.17)

$$2i(A_t + \tfrac{1}{2} A_x) + \tfrac{1}{2} A_{yy} - \tfrac{1}{4} A_{xx} - A|A|^2 = - \frac{1}{8} i (6A_{xyy} - A_{xxx}) +$$

$$+ \frac{3}{2} i A (AA_x^* - A^* A_x) - \tfrac{1}{2} |A|^2 A_x + A(\bar{\phi}_x - i\bar{\phi}_z) \qquad \text{(Dysthe 2.19)}$$

2.2 Third Order Evolution Equations

Equations (2.1.32) and (2.1.55) constitute the following third order set of evolution equations:

$$\mu_1 A + A_2 A_\xi + \mu_3 A_{\tau\tau} + \mu_4 A\phi_{10_\tau} + \mu_5 |A|^2 A + \mu_6 A\phi_{20} = 0 \qquad (2.2.1)$$

and

$$\phi_{10_z} + \frac{\varepsilon^2 \phi_{10_{\tau\tau}}}{g} = \frac{\varepsilon^2 k^2}{g}\left[\frac{2C_p}{C_g} + (1-\sigma^2)\right] (|A|^2)_\tau \qquad (2.2.2)$$

Shallow water

For shallow water, we assume that the horizontal induced current is uniformly distributed with depth; and then integrating equation (2.1.1) yields

$$\phi_{10_z} = \frac{-\varepsilon^2 \phi_{10_{\tau\tau}}}{C_g^2} (z+h) \qquad (2.2.3)$$

Substituting (2.2.3) for z=0 in (2.2.2) yields

$$\phi_{10_{\tau\tau}} \left(-\frac{h}{C_g^2} + \frac{1}{g}\right) = \varepsilon \frac{k^2}{g}\left[\frac{2C_p}{C_g} + (1-\sigma^2)\right](|A|^2)_\tau \qquad (2.2.4)$$

integrating (2.2.4) with respect to τ gives

$$\phi_{10_\tau} = \frac{k^2}{\left(1 - \frac{gh}{C_g^2}\right)} \left[\frac{2C_p}{C_g} + (1-\sigma^2)\right] |A|^2 + Q(\xi) \qquad (2.2.5)$$

The set (2.2.1) and (2.2.5) is the same as that given by Djordjevic' and Redekopp (1978), except the term $\mu_6 A\phi_{20}(\xi)$ which does not appear in Djordjevic and Redekopp.

In the sequel we will show the necessity of including the term $\varepsilon^2\phi_{20}(\xi)t$ in the wave-induced mean current expansion (see (2.1.11).

To second order in ε the wave induced mean current velocity is given by

$$U = \varepsilon\phi_{10_x} = \frac{\varepsilon^2\ k^2}{C_g(1-\frac{gh}{C_g^2})}\left[\frac{2C_p}{C_g} + (1-\sigma^2)\right]|A|^2 + \frac{\varepsilon^2 Q(\xi)}{C_g} \qquad (2.2.6)$$

To find Q we introduce a lateral boundary condition of zero averaged (over τ) mass flow which is appropriate for an impervious beach, as follows:

$$h\ \bar{U} + \varepsilon^2 k\,\frac{2\omega}{g}\,\overline{|A|^2} \qquad (2.2.7)$$

where the bar indicates the averaging.

From eqs. (2.2.7) and (2.2.6) we obtain

$$Q(\xi) = -\overline{|A|^2}\left\{\frac{2\omega k C_g}{gh} + \frac{k^2}{(1-\frac{gh}{C_g})}\left[\frac{2C_p}{C_g^2} + (1-\sigma^2)\right]\right\} \qquad (2.2.8)$$

Integration of eq. (2.2.5) with respect to τ yields

$$\phi_{10} = \frac{k^2}{(1-\frac{gh}{C_g^2})}\left[\frac{2C_p}{C_g} + (1-\sigma^2)\right]\int |A|^2\, d\tau + Q(\xi)\tau + \varepsilon Q_2(\xi) \qquad (2.2.9)$$

The first and second terms in (2.2.9) grow monotinically, and boundlessley in time. Secular terms of this nature are bound to cause trouble in higher order derivation and should be suppressed. The addition of the term $\varepsilon^2\phi_{20}(\xi)t$, to eq. (2.1.11) seems to be the proper way to achieve this goal. Substituting eq. (2.2.9) into (2.1.11) gives

$$\phi_o = \varepsilon^2 \frac{k^2}{(1-\frac{gh}{C_g^2})} [\frac{2C_p}{C_g} + (1-\sigma^2)] \{ \int_{x_o}^{x} \frac{|A|^2 dx}{C_g} - \int_0^t (|A|^2 - \overline{|A|^2}) dt - \overline{|A|^2} t\}$$

$$+ \varepsilon^2 Q \{\int_{x_o}^{x} \frac{dx}{C_g} - t\} + \varepsilon^2 Q_2(\xi) + \varepsilon^2 \phi_{20}(\xi) t \qquad (2.2.10)$$

Thus suppressing secular terms in t we get

$$\phi_{20}(\xi) = \frac{k^2[\frac{2C_p}{C_g} + (1-\sigma^2)]}{(1-\frac{gh}{C_g^2})} + Q = -\frac{2\omega k C_g}{gh} \overline{|A|^2} \qquad (2.2.11)$$

Finally, substitution of eq. (2.2.8) for Q, and (2.2.11) for $\phi_{20}(\xi)$ into equation (2.2.1); and introduction of the new variable $\tilde{\psi} = \frac{2\omega}{g} \varepsilon A$ lead to the following nonlinear Schrödinger equation for the wave envelope $\tilde{\psi}$.

$$\frac{i}{2C_g} \frac{\partial C_g}{\partial \xi} \tilde{\psi} + i\tilde{\psi}_\xi + \frac{C_g''}{2C_g^3} \tilde{\psi}_{\tau\tau} - \frac{\varepsilon^2 \alpha_1}{C_g} |\tilde{\psi}|^2 \tilde{\psi} =$$

$$= \frac{\alpha_2}{C_g} \varepsilon^{-2} \overline{a^2} \tilde{\psi} \qquad (2.2.12)$$

where

$$\alpha_1 = \frac{gk^3}{2\omega} \cdot \frac{9th^4(kh) - 10th^2(kh) + 9}{8th^3(kh)} - (\frac{g^2k}{2\omega} + \frac{gkC_g}{2sh(2kh)}) \cdot \frac{k}{gh-(C_g)^2} +$$

$$- (\frac{gkh}{2sh(2kh)} + \frac{gkC_g}{2\omega}) \cdot \frac{gk^2}{2\omega[gh-(C_g)^2] ch^2(kh)} \qquad (2.2.13)$$

$$\alpha_2 = -\alpha_1 - \frac{gk^2}{2\omega h} - \frac{gk^3}{4\omega sh(2kh) ch^2(kh)} + \frac{gk^3}{2\omega} \cdot$$

$$\cdot \frac{9th^4(kh) - 10th^2(kh) + 9}{8th^3(kh)} \qquad (2.2.14)$$

where $a = |\tilde{\psi}|$ (2.2.15)

Equation (2.2.12) will be rederived in section (2.3) by means of the Whitham equations.

2.3 Third Order Equations Obtained from Whitham Equations

We consider here the case of shallow water and recover the nonlinear Schrödinger equation (2.2.1) by using Whitham's modulation equations.

(2.3a) Modulation equations

Considering the 2-D problem of wave-groups propagating over water of slowly varying depth, the following five unknowns are usually chosen as dependent variables: the wave amplitude a, the wave frequency $\tilde{\omega}$, the wave number $\tilde{k}$, the average water depth $\tilde{h}$, and the current velocity U. To determine these unknowns we start from Whitham's set of modulation equations, Whitham (1974), p. 556. Pseudo-phase consistency relation:

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left[g(\tilde{h}-h) + \frac{g\tilde{k}}{2sh(2\tilde{k}\tilde{h})}\, a^2 + 0(\varepsilon^4)\right] = 0 \qquad (2.3a.1)$$

Mass conservation equation:

$$\frac{\partial(\tilde{h}-h)}{\partial t} + \frac{\partial}{\partial x}\left[h\, U + \frac{g\tilde{k}}{2\tilde{\sigma}}\, a^2 + 0(\varepsilon^4)\right] = 0 \qquad (2.3a.2)$$

Wave-action conservation equation:

$$\frac{\partial}{\partial t}\left[\frac{a^2}{\tilde{\sigma}} + O(\varepsilon^4)\right] + \frac{\partial}{\partial x}\left[\tilde{\sigma}' \frac{a^2}{\tilde{\sigma}} + O(\varepsilon^4)\right] = 0 \qquad (2.3a.3)$$

Consistency condition:

$$\frac{\partial \tilde{k}}{\partial t} + \frac{\partial \tilde{\omega}}{\partial x} = 0 \qquad (2.3a.4)$$

where $\tilde{\sigma} = [g\tilde{k}\ \mathrm{th}(\tilde{k}h)]^{\frac{1}{2}}$ is the linear dispersion relation, $\tilde{\sigma}' = \partial\tilde{\sigma}/\partial\tilde{k}$, and ε is a typical wave steepness, $\varepsilon = O(ak)$.

Following Whitham (1974), p. 562 and including higher order dispersive terms, see Whitham p. 526, (which arise from the quadratic part of the Lagrangian $L = \tilde{\omega} - \tilde{\sigma}$), the dispersion relation is given by:

$$\tilde{\omega} = \tilde{\sigma} + \tilde{k}U + \frac{g\tilde{k}^2}{2\tilde{\sigma}\,\mathrm{ch}^2(\tilde{k}h)}(\tilde{h}-h) + \frac{g\tilde{k}^3 C}{2\tilde{\sigma}} a^2 - \frac{\tilde{\sigma}''}{2a}\frac{\partial^2 a}{\partial x^2} + O(\varepsilon^4) \qquad (2.3a.5)$$

here $C = (9\mathrm{th}^4(\tilde{k}h) - 10\mathrm{th}^2(\tilde{k}h) + 9)/8\mathrm{th}^3(\tilde{k}h)$

Here, and in what follows, we assume that the small modulation parameter ε is of the same order of magnitude as the typical wave steepness. We also assume, as we did in section (2.1), very mild depth changes, having slopes of the order of ε^2 at most.

(2.3b) Induced Mean Flow

To make the slow variation explicit and to facilitate the derivation we introduce the same multiple scale variables, τ and ξ as given in section (2.1) by (2.1.5).

Rewriting eqs. (2.3a.3) and (2.3a.4) with the new coordinates (2.1.5) and averaging them over τ gives:

$$\overline{\frac{\tilde{g}'}{\tilde{\sigma}} a^2} = \text{const}_1 = B; \qquad \bar{\tilde{\omega}} = \text{const}_2 = \omega \tag{2.3b.1}$$

Here we assume that the behavior of the solution as a function of τ is the same as that of the linear boundary condition at x_∞. Namely, decaying for $|\tau| \to \infty$ in the case of wave-packets and with a constant period in the case of modulated wave-trains.

Again, bars indicate averaging over the appropriate domain in τ (finite for modulated wave-trains and infinite for wave packets) and B, ω are the averaged wave-action flux and the so-called carrier frequency, respectively. Note that for wave-packets B = 0.

The carrier wave number k, satisfies the dispersion relation (2.1.25).

Rewriting eqs. (2.3a.1) and (2.3a.2) with the new independent variables, yields:

$$\frac{\partial}{\partial\tau} \{\frac{1}{C_g} [g(\tilde{h}-h)+ \frac{g\tilde{k}}{2sh(2\tilde{k}h)} a^2] - U\} + \varepsilon \frac{\partial}{\partial\xi} \{g(\tilde{h}-h) + $$

$$+ \frac{g\tilde{k}a^2}{2sh(2\tilde{k}h)} = 0 \tag{2.3b.2}$$

$$\frac{\partial}{\partial\tau} \{\frac{1}{C_g} [h\, U + \frac{g\tilde{k}}{2\tilde{\sigma}} a^2] - (\tilde{h}-h)\} + \varepsilon \frac{\partial}{\partial\xi} \{h\, U + \frac{g\tilde{k}}{2\tilde{\sigma}} a^2\} = 0 \tag{2.3b.3}$$

Neglecting the second terms in the above equations for the time being, we obtain:

$$\tilde{h}-h = -\left(\frac{g\tilde{k}h}{2sh(2\tilde{k}h)} + \frac{g\tilde{k}C_g}{2\tilde{\sigma}}\right)\frac{a^2}{gh-(C_g)^2} + \frac{h\ G_1+C_g G_2}{gh-(C_g)^2} \tag{2.3b.4}$$

$$U = -\left(\frac{g^2\tilde{k}}{2\sigma} + \frac{g\tilde{k}C_g}{2sh(2\tilde{k}h)}\right)\frac{a'}{gh-(C_g)^2} + \frac{gG_2+C_g G_1}{gh+(C_g)^2} \tag{2.3b.5}$$

where G_1 and G_2 are functions of ξ, which emerged as a result of the integration. Now, everaging eqs. (2.3b.2) and (2.3b.3) and substituting (2.3b.4) and (2.3b.5) yields:

$$\left[\frac{gk}{2sh(2kh)} - \frac{g}{gh-(C_g)^2}\left(\frac{gkh}{2sh(2kh)} + \frac{gkC_g}{2\omega}\right)\right]\overline{a^2} + $$

$$+gC_g\frac{h\ G_1+C_g G_2}{gh-(C_g)^2} = \text{const}_3 \tag{2.3b.6}$$

$$\left[\frac{gk}{2} - \frac{h}{gh-C_g^{\,2}}\left(\frac{g^2k}{2\omega} + \frac{gkC_g}{2sh(2kh)}\right)\right]\overline{a^2} + hC_g\frac{gG_2+C_g G_1}{gh-(C_g)^2} = \text{const}_4 \tag{2.3b.7}$$

where, consistent with the order of approximation $\tilde{k}$ and $\tilde{\sigma}$ have been replaced by k,ω.

Following Stiassnie and Peregrine (1980) we assume zero averaged mass flow (thus, $\text{const}_4=0$) and choose such a reference level that $\overline{\tilde{h}-h} = 0$ in deep water (which, in turn, sets $\text{const}_3=0$). Having fixed these two constants we solve eqs. (2.3b.6) and (2.3b.7) for G_1 and G_2 and then return to eqs.(2.3b.4), (2.3b.5) to obtain the final results for the induced mean flow:

$$U(\tau,\xi) = \left(\frac{g^2k}{2} + \frac{gkC_g}{2sh(2kh)}\right) \cdot \frac{\overline{a^2} - a^2}{gh-(C_g)^2} - \frac{gk\overline{a^2}}{2\omega h} \tag{2.3b.8}$$

$$\tilde{h}(\tau,\xi)-h(\xi) = \left(\frac{gkh}{2sh(2kh)} + \frac{gkC_g}{2}\right) \cdot \frac{\overline{a^2} - a^2}{gh-(C_g)^2} - \frac{k\overline{a^2}}{2sh(2kh)} \tag{2.3b.9}$$

Consistent with our level of approximation we have

$$\overline{a^2} = \frac{\omega}{C_g} B = \begin{cases} \dfrac{g}{2\omega C_g} \overline{a^2_\infty}, & \text{for modulated wave-trains} \\ 0, & \text{for wave-packets} \end{cases} \tag{2.3b.10}$$

(2.3c) Nonlinear Schrödinger Equation

To derive the NLS we reqrite eq. (2.3a.3) as follows

$$\frac{2a}{\tilde{\sigma}} \frac{\partial a}{\partial t} - \frac{\tilde{\sigma}' a^2}{\tilde{\sigma}} \frac{\partial \tilde{k}}{\partial t} + \frac{2\tilde{\sigma}' a}{\tilde{\sigma}} \frac{\partial a}{\partial x} - \frac{\tilde{\sigma}' a^2}{\tilde{\sigma}^2} \frac{\partial \tilde{\sigma}}{\partial x} + \frac{a^2}{\tilde{\sigma}} \frac{\partial \tilde{\sigma}'}{\partial x} = 0 \tag{2.3c.1}$$

Applying eq. (2.3a.4), which gives mutual cancellation of the second and fourth terms in the above equation, and dividing by $2a/\tilde{\sigma}$ yields

$$\frac{\partial a}{\partial t} + \tilde{\sigma}' \frac{\partial a}{\partial x} + \frac{a}{2} \frac{\partial \tilde{\sigma}'}{\partial x} = 0 \tag{2.3c.2}$$

The Taylor series expansion of $\tilde{\sigma}(\tilde{k})$ in the vicinity of $\tilde{k} = k$ is

$$\tilde{\sigma} = \omega + C_g \cdot (\tilde{k}-k) + \frac{\omega''}{2} \cdot (\tilde{k}-k)^2 + O(\varepsilon^2) \tag{2.3c.3}$$

Using this series we rewrite eq. (2.3c.2) as well as the dispersion relation (2.3a.5):

$$\frac{\partial a}{\partial t} + C_g\frac{\partial a}{\partial x} + \omega''\cdot(\tilde{k}-k)\frac{\partial a}{\partial x} + \frac{\omega'' a}{2}\frac{\partial(\tilde{k}-k)}{\partial x} + \frac{a}{2}\frac{\partial C_g}{\partial x} = 0 \qquad (2.3c.4)$$

$$\tilde{\omega} = \omega + C_g \cdot (\tilde{k} - k) + \frac{\omega''}{2}\cdot(\tilde{k} - k)^2 + \alpha_1 a^2 - \frac{\omega''}{2a}\frac{\partial^2 a}{\partial x^2} + \alpha_2\overline{a^2} \qquad (2.3c.5)$$

where

α_1 and α_2 are given in section (2.3) by eqs. (2.2.13) and (2.2.14) respectively.

Referring to eq. (2.3a.4) we define a phase function φ so that

$$\tilde{\omega} - \omega = -\frac{\partial\varphi}{\partial t}, \qquad \tilde{k} - k = \frac{\partial\varphi}{\partial x} \qquad (2.3c.6)$$

Substituting eqs. (2.3c.6) into (2.3c.4) and (2.3c.5) we obtain

$$\frac{\partial a}{\partial t} + C_g\frac{\partial a}{\partial x} + \frac{\omega''}{2}\left(a\frac{\partial^2\varphi}{\partial x^2} + 2\frac{\partial\varphi}{\partial x}\frac{\partial a}{\partial x}\right) + \frac{a}{2}\frac{\partial C_g}{\partial x} = 0 \qquad (2.3c.7)$$

$$\frac{\partial\varphi}{\partial t} + C_g\frac{\partial\varphi}{\partial x} + \frac{\omega''}{2}\left(\left(\frac{\partial\varphi}{\partial x}\right)^2 - \frac{1}{a}\frac{\partial^2 a}{\partial x^2}\right) + \alpha_1 a^2 + \alpha_2\overline{a^2} = 0 \qquad (2.3c.8)$$

Multiplying eq. (2.3c.7) by the imaginary unit i, adding to it (-a) times eq. (2.3c.8), and then multiplying the sum by $e^{i\varphi}$ we get

$$\frac{i}{2}\cdot\frac{\partial C_g}{\partial x}\cdot\tilde{\psi} + i(\tilde{\psi}_t + C_g\tilde{\psi}_x) + \frac{\omega''}{2}\tilde{\psi}_{xx} - \alpha_1|\tilde{\psi}|^2\tilde{\psi} = \alpha_2\overline{a^2}\,\tilde{\psi} \qquad (2.3c.9)$$

where $\tilde{\psi} = a\,e^{i\varphi}$ is a complex wave envelope. Alternatively, using the scaled coordinates (2.1.5) we obtain the N.L.S. equation (2.2.12).

3. SOLUTIONS

In this chapter we solve the third order evolution equations for two particular important cases. The relevant eqs. were derived in chapter 2, and are given here again for the sake of clarity.

From the Laplace's equation (2.1.1) it follows that

$$\phi_{10_{zz}} + \frac{\varepsilon^2}{C_g^2}\phi_{10_{\tau\tau}} = 0 \qquad -h \leqslant z \leqslant 0 \tag{3.1}$$

From the boundary condition at the bottom (2.1.2) we obtain

$$\phi_{10_z} = 0, \qquad z = -h \tag{3.2}$$

The boundary conditions at z=0 are

$$\phi_{10_z} + \frac{\varepsilon^2}{g}\phi_{10_{\tau\tau}} = \frac{g\beta_2}{2\omega}(|\tilde{\psi}|^2)_\tau \quad ; \qquad z = 0 \tag{3.3}$$

(same ase (2.2.2))

$$\frac{i}{2C_g}\cdot\frac{\partial C_g}{\partial\xi}\tilde{\psi} + i\tilde{\psi}_\xi + \frac{\omega''}{2(C_g)^3}\tilde{\psi}_{\tau\tau} - \frac{\varepsilon^{-2}\beta_1}{C_g}|\tilde{\psi}|^2\tilde{\psi} =$$

$$= (\beta_2\phi_{10_\tau} + \beta_3\phi_{20})\frac{\tilde{\psi}}{C_g}; \qquad z = 0 \tag{3.4}$$

(same as (2.2.1))

where

$$\beta_1 = \frac{gk^3}{2\omega}\cdot\frac{9-12\sigma^2 + 13\sigma^4 - 2\sigma^6}{8\sigma^3} \tag{3.5}$$

$$\beta_2 = \frac{k^2}{2\omega}\cdot\left(\frac{2\omega}{kC_g} + (1-\sigma^2)\right) \tag{3.6}$$

and

$$\beta_3 = \frac{k^2}{2\omega}(1-\sigma^2) \tag{3.7}$$

In section (3.1) we consider the case of periodical boundary conditions, and present an approximate analytical solution for the shoaling of wave groups over a very slowly varying topography. In section (3.2) we consider the case of infinitely deep water and calculate the induced mean flow accompanying an envelope soliton.

(3.1) Shoaling of wave-groups

Restricting the discussion to cases for which the complex wave envelope $\tilde{\psi}(\tau,\xi)$ is periodic in τ, and assuming zero averaged (over τ) mass flow in the x direction, enables the decoupling of equations (3.3) and (3.4) (see Appendix A). For this case $\tilde{\psi}$ is governed by the nonlinear Schrödinger equation (2.2.12):

$$\frac{i}{2C_g}\frac{\partial C_g}{\partial \xi}\tilde{\psi} + i\tilde{\psi}_\xi + \frac{C_g''}{2C_g^3}\tilde{\psi}_{\tau\tau} - \frac{\varepsilon^{-2}\alpha_1}{C_g}|\tilde{\psi}|^2\tilde{\psi} =$$

$$= \frac{\alpha_2}{C_g}\varepsilon^{-2}\overline{a^2}\,\tilde{\psi}$$

with α_1 and α_2 given by (2.2.13), and (2.2.14) respectively. The induced mean flow potential is given by

$$\phi_{10} = -\frac{\varepsilon^{-2}\overline{|\tilde{\psi}|^2}gC_g k}{2\omega h}\tau + \sum_{n=1}^{\infty} a_n(z)\{\, b_n(\xi)\, e^{\frac{2\pi i n\tau}{\gamma}} +$$

$$+ b_{-n}(\xi)\, e^{\frac{-2\pi i n\tau}{\gamma}} + Q_2(\xi) \tag{3.1.1}$$

where γ is the period of $\tilde{\psi}(\xi,\tau)$.

$$a_n = \cosh(2\pi\varepsilon n(z+h)/C_g\gamma) \qquad n=1,2,\ldots,; \tag{3.1.2}$$

$$b_n = \frac{\varepsilon^{-2} g^2 \beta_2}{2\omega\{ \frac{2\pi n g}{\varepsilon C_g \gamma} \sinh(\frac{2\pi n \varepsilon h}{C_g \gamma}) - (\frac{2\pi n}{\gamma})^2 \cosh(\frac{2\pi n \varepsilon h}{C_g \gamma})} \cdot C_n \tag{3.1.3}$$

and C_n are the Fourier coefficients of $(|\tilde{\psi}|^2)_\tau$:

$$(|\tilde{\psi}|^2)_\tau = \sum_{n=1}^{\infty} \{C_n(\xi) e^{\frac{2\pi i n \tau}{\gamma}} + C_{-n}(\xi)\, e^{-\frac{2\pi i n \tau}{\gamma}} \tag{3.1.4}$$

The convergence of the Fourier series (3.1.4) was assumed to be independent of ε .

The potential ϕ_{20} which is needed to calculate the mean water level ζ_o, see eq. (2.1.27) is given by

$$\phi_{20} = -\varepsilon^{-2} \overline{|\tilde{\psi}|^2} \frac{g\, k\, C_g}{2\omega h} \tag{3.1.5}$$

All technical details are given in Appendix 3.

A simpler and dimensionless form of eq. (2.2.12) is obtained by means of the transformations

$$\psi = \varepsilon^{-1} \left(\frac{2\omega^5 C_g}{g^3}\right)^{\frac{1}{2}} \tilde{\psi} \exp i \left(\overline{|\tilde{\psi}|^2} \int_{x_\infty}^{x} \frac{\alpha_2 dx}{C_g}\right); \tag{3.1.6}$$

$$T = \tau/\gamma; \qquad X = \frac{1}{\gamma^2}\int_{\xi_\infty}^{\xi} \frac{\omega''}{2(C_g)^3}\, d\xi \tag{3.1.7}$$

which give

$$i\psi_X + \psi_{TT} + \mu|\psi|^2\psi = 0, \tag{3.1.8}$$

$$\mu(\xi) = \frac{-g^3 C_g \gamma^2 \alpha_1}{\omega^5 \omega''} \tag{3.1.9}$$

The dimensionless parameter μ is a monotonic increasing function of kh, having the values zero and $(\omega\gamma)^2$ for kh=1.363 and kh $\to \infty$ respectively. The statement of the mathematical problem, given by Eq. (3.1.8), is completed by the following input condition at X = 0. (i.e. $x = x_\infty$; a reference point in infinitely deep water).

$$\psi(T,0) = 1 + 2\beta e^{i\alpha}\cos(2\pi T) \tag{3.1.10}$$

which corresponds to a system composed of a carrier-wave and a symmetric "side-band" disturbance.

$$\zeta(t,x_\infty) = \frac{g\varepsilon}{\omega^2}\,\mathrm{Re}\{e^{-i\omega t} + \beta e^{-i[(1+2\pi\varepsilon/\omega\gamma)\omega t-\alpha]} +$$

$$+ \beta e^{-i[(1-2\pi\varepsilon/\omega\gamma)\omega t-\alpha]}\} + O(\varepsilon^2) \tag{3.1.11}$$

<u>For constant depth</u>, μ = const, it is well-known that Eq. (3.1.8) with T in (0,1), subject to periodic boundary conditions has the following X invariants

$$J_1 = \int_0^1 |\psi|^2 dT \tag{3.1.12}$$

$$J_2 = \int_0^1 (\psi^*\psi_T - \psi\psi_T^*)\, dT \tag{3.1.13}$$

$$J_3 = \int_0^1 (|\psi|^4 - \frac{2}{\mu}|\psi_T|^2)\, dT \tag{3.1.14}$$

These invariants are determined by the input condition, Eq. (3.9) so that

$$J_1 = 1 + 2\beta^2; \qquad J_2 = 0 \tag{3.1.15}$$

$$J_3 = 1 + (4-P+2\cos 2\alpha)\cdot 2\beta^2 + 6\beta^4 \tag{3.1.16}$$

where

$$P = 8\pi^2/\mu \tag{3.1.17}$$

For varying depth, $\mu = \mu(X)$, J_1 and J_2 remain invariant and are given by (3.1.12), (3.1.13) and (3.1.15), (3.1.16), but J_3 is a function of X governed by the equation,

$$\frac{dJ_3}{dX} = \frac{2\mu_X}{\mu^2}\int_o^1 |\psi_T|^2 dT \tag{3.1.18}$$

(See Appendix 5).

(3.1a) Three-Waves systems

The solution of Eq. (3.1.8) can be expanded in a Fourier series

$$\psi(T,X) = \sum_{n=-\infty}^{\infty} D_n(X)e^{2\pi inT} \tag{3.1a.1}$$

The boundary condition at X = 0, Eq. (3.1.10), gives $D_o(0)=1$; $D_1(0) = D_{-1}(0) = \beta e^{i\alpha}$; $D_n(0) = D_{-n}(0) = 0$ for $n \geqslant 2$.

Stiassnie and Kroszynski (1982) truncated the above given series and considered only three waves systems:

$$\psi(T,X) = \sum_{n=-1}^{1} D_n(X)e^{2\pi inT} \tag{3.1a.2}$$

Substituting Eq. (3.1a.2) into Eq. (3.1.8) yields the following system of ordinary differential equations:

$$i \frac{dD_o}{dX} + \mu[(|D_o|^2+4|D_1|^2)D_o+2D_1^2D_o^*] = 0, \tag{3.1a.3}$$

$$i \frac{dD_1}{dX} + \mu[(2|D_o|^2+3|D_1|^2-\frac{P}{2})D_1 + D_o^2D_1^*] = 0. \tag{3.1a.4}$$

Note that Eq. (3.1.13) yields $D_{-1} = D_1$.

For constant depth the system of Eqs. (3.1a.3), (3.1a.4) has exact solutions in terms of Jacobian elliptic functions with periods of order 1 in X which is summarized in Appendix B; for details see Stiassnie and Kroszynski (1982). These solutions depend on the invariants J_1, J_3 and on the parameter μ , which in turn depends on the water depth h and on the modulation period γ . For very mild depth variations, where $h_X = o(1)$, we apply an asymptotic, WKB related approach, assuming the local solution to be that of the constant depth type and using Eq. (3.1.18) to determine J_3. J_1 and γ are fixed by the input conditions and J_3, is given through I(P) by:

$$\frac{dI}{dP} = -\sqrt{\frac{P(4-P)}{7}} \; \frac{2\cdot\ln\left(\dfrac{1+\sqrt{\frac{4-P}{7P}}}{1-\sqrt{\frac{4-P}{7P}}}\right)}{\ln\left(\dfrac{2P^2(4-P)^2}{|(2P-1)I|}\right)} \tag{3.1a.5}$$

(see Appendix 4), where

$$I(P) = J_3(P) - J_1^2 \tag{3.1a.6}$$

The initial value of I, at X = 0, where $P = P_o$ is denoted by I_o and is given by

$$I_o = \beta^2[2\beta^2 + 4(1+\cos 2\alpha)-2P_o] \tag{3.1a.7}$$

I_o, as well as I(P) were assumed to be of o(1) throughout the rather lengthy derivation of Eq. (3.1a.5). In all our examples we choose ($\gamma = 2\pi\omega^{-1}$) P_o=2, corresponding to the fastest growth-rate of the Benjamin-Feir instability.

(3.1b) Numerical Verification of the asymptotic solution

In order to appraise the relevance of the asymptotic solution given in the previous section, we compare it's results with those of a numerical solution of the system of ordinary differential equations (3.1a.3), (3.1a.4).

Fig. 1 shows I=I(P) for four initial values of I_o=I(2) (I_o= -0.04, -0.01, 0.04 and 0.1). The broken line represents the asymptotic solution and was obtained by numerical integration of Eq. (3.1a.5). The solid line was obtained, by substitution of the results obtained from a numerical solution of the system of O.D.E. (3.1a.3),(3.1a.4) into the expression

$$I(P) = 2|D_1|^2\{|D_1|^2+2|D_o|^2-P+2|D_o|^2\cos[2(\arg D_1-\arg D_o)]\} \quad (3.1b.1)$$

The numerical solution of the system of O.D.E was obtained using a trapezoidal method and assuming the P(h(X)) = 2+0.2X. Note that the assumption $|I| \ll 1$, which is necessary for the asymptotic solution to the valid, imposes a restriction on the range of variation of P (P=2.8 corresponds to $\omega^2 h/g=4$).

In Fig. 2 we show three parts of the exterior group envelope $|\psi(0,X)|$ as well as the interior group envelope $|\psi(\frac{1}{2},X)|$ for the input conditions $\alpha = 0$, $\beta = 0.158$ ($I_o = 0.1$).

Here again, solid lines represent the numerical solution of Eqs. (3.1a.3) (3.1a.4) with $P = 2+0.2X$ while the broken lines correspond to results obtained by the asymptotic method, utilizing the relation:

$$|\psi(T,X)|^2 = \{-2I+2(4J_1-P)\tilde{z}-7\tilde{z}^2+[S(2I+2P\tilde{z}-\tilde{z}^2)^{\frac{1}{2}} +$$

$$+ 4\tilde{z}\cos 2\pi T]^2\}/8\tilde{z} \qquad (3.1b.2)$$

where $\tilde{z}$ is given in Appendix B.

The thre parts shown in Fig. 2 are for P =2, 2.44, 2.75 for the asymptotic solution compared to P in (2,2.03), (2.44,2.49), (2.75, 2.81) for the numerical solution of the O.D.E respectively. The agreement between the two methods of solution, as seen in both the above figures is rather encouraging and seems to indicate the validity of our new asymptotic solution of the system (3.1a.3)(3.1a.4) Nevertheless, one still has to answer the question if, and to what extent, the system (3.1a.3,4) itself is a reasonable substitute for the N.L.S. equation. For constant depth Stiassnie and Kroszynski (1982) show a good quantitative agreement in the length of the modulation-demodulation cycle and only a qualitative agreement for the amplitudes. A similar trend can be seen in Fig. 3, which compares to numerical solutions, for for the N.L.S. (3.1.8) - dotted line, and the other for the system of O.D.E. (3.1a,3,4) - solid line.

The input data in Fig. 3 is $\alpha = 0$, $\beta = 0.1$ ($I_o = 0.04$) and the variation $P = 2 + 0.2X$ is assumed; Both the exterior and interior group envelopes are drawn.

We believe that our much-simplified asymptotic solution is not over-simplified, and is able to produce quite a few results of qualitative, and maybe even semi-quantitative relevance, which enable us some new physical insight.

(3.1c) On P and I_o

One fundamental property of the asymptotic solution is that it depends on P and I_o solely. Given the input data α, β (and P_o=2), I_o is determined by Eq. (3.1a.7). Then, integrating Eq. (3.1a.5) from P_o to P the parameter I(P) is found, and the solution given by Stiassnie and Kroszynski (1982), is locally applied.

Fig. 4 gives the relation between P and nondimensional local water depth $k_\infty h$, (where $k_\infty = \omega^2/g$ is the wave-number in infinitely deep water), for $\gamma = 2\pi\omega^{-1}$. P is a monotonic decreasing function having the values infinity at $k_\infty h = 1.195$ (kh=1.363) and 2 for $k_\infty h \to \infty$. The input value I_o, (for P_o=2) dependence on α and β is shown in Fig. 5. Note that different combinations of α and β give the same I_o, and thus basicly, the same solution for any P.

(3.1d) Group envelopes

The free-surface of an (unstable) shoaling wave-train, displays three distinct length scales: ℓ_1 - the wave length; ℓ_2 - the modulation or group length; and ℓ_3 - the modulation-demodulation, group-envelope, or maybe best 'supergroup' length. These three lengths are given by

$$\ell_1 = 2\pi/k \tag{3.1d.1}$$

$$\ell_2 = 2\pi\varepsilon^{-1} C_g/\omega. \tag{3.1d.2}$$

$$\ell_3 = \frac{4\varepsilon^{-2} P (C_g)^3}{\omega^2 \omega'' \sqrt{P(4-P)}} \ln \left| \frac{2P^2 (4-P)^2}{(2P-1) I} \right| , \tag{3.1d.3}$$

see Fig. 7a .

It can easily be seen that in the range of depths where the asymptotic solution applies, $k_\infty h > 4$, ℓ_1 and ℓ_2 remain almost constant. On the other hand, ℓ_3, which depends on P, exhibits quite a remarkable variation as shown in Fig. 6 .

In Fig. 5 , we show the variation of ℓ_3 as a function of the depth $k_\infty h$ for four different input data I_o=-0.04, -0.01, 0.04, 0.1. For $I_o < 0$, ℓ_3 decreases with decreasing depth, but for $I_o > 0$ ℓ_3 increases with decreasing depth up to a "critical depth" (corresponding to I = 0) and from there on starts to decrease.

Fig. 7 shows the group envelopes (dashed line) and wave envelope (solid lines) at a fixed instant for ε = 0.2, at the following four locations: (a) - infinitely deep water, $P_o = 2$, $I_o = 0.1$; (b) $k_\infty h = 11.2$, $P = 2.2$, $I = 0.052$; (c) $k_\infty h = 5.7$, $P = 2.45$, $I = 0.01$; (d) $k_\infty h = 4.2$, $P = 2.75$, $I = -0.028$.

In Fig. 7 a we have added a portion of the wavy-surface (thin solid line) as well as the lengths ℓ_1, ℓ_2 and ℓ_3. Note that the super-groups (namely: the exterior and interior group envelopes) are fixed in space, while the wave envelope moves with group-velocity and the waves themselves with the phase velocity. Similar sketches to Fig. 7 were obtained for the other cases given in Fig. 6 .

In order to complete the picture for shallower water depth we present in Fig. 8 the group envelopes at five locations: P = (2,2.05), b:(2.32, 2.63); c:(17.7, 22.8), d:(-22.4, -17.5), e:(-4.1,-3.9), as obtained from a numerical solution of the system (4.3a,b) for the same input data as in Fig. 7 assuming $\mu = 39.47-10X$.

The results in Fig. 8 indicate that ℓ_3 continues to shorten and that the intensity of modulation decreases.

(3.1e) The mean flow field

We express the mean flow $U = \partial\phi_o/\partial x = -\partial\Psi/\partial z$, $V = \partial\phi_o/\partial z = \partial\Psi/\partial x$ through the stream function Ψ

$$\Psi = \frac{g^2 k\gamma\varepsilon^{-1}}{4\pi\omega^3 h}\left(|D_o|^2+2|D_1|^2\right)Z + \frac{g^3\gamma\varepsilon^{-1}}{4\pi\omega^3 C_g}\cdot$$

$$\cdot\frac{\beta_2(D_o^*D_1+D_oD_1^*)\,sh(2(Z+H))\cos(2\pi T)}{\frac{g\gamma}{2\pi\varepsilon C_g}\,sh(2H)-ch(2H)} + \frac{g^2\gamma\varepsilon^{-1}}{4\pi\omega^3 C_g}\cdot$$

$$\cdot\frac{\beta_2|D_1|^2 sh(4(z+H))\cos(4\pi T)}{\frac{g\gamma}{2\pi\varepsilon C_g}\,sh(4H)-2ch(4H)} + \text{constant} \qquad (3.1e.1)$$

where $Z = \pi\varepsilon z/C_g\gamma$, $H = \pi\varepsilon h/C_g\gamma$ and $\tilde{\Psi} = \varepsilon^{-2}\omega^3\Psi/g^2$ are dimensionless quantities. The constant in Eq. (3.1e.1) is chosen so that $\Psi = 0$ at the bottom. The mean free surface ζ_o is given by Eqs. (2.1.16) and (2.1.27).

The stream-function $\tilde{\Psi}(T,Z)$ as well as the mean free-surface for cases a, b and d of Fig. 7 are presented in Figures 9 , 10 and 11 respectively. These figures demonstrate the rather complicated structure of the wave induced mean flow field.

Some of the main features are: (i) the mean current, which is shown in part (b) of the figures, as well as the mean free surface, in part (c) exhibit a somewhat cellular structure influenced by the wave envelope variations; (ii) a dominant adverse current appears underneath the high waves and a much weaker, positive current (in the wave propagation direction)under the low waves, for the shallower cases the positive currents almost disappears; (iii) the magnitude of the maximum adverse currents at the free surface is almost the same for all three depths ($k_\infty h = \infty$, 11.2 and 4.2); (iv) one can notice the tendency of the flow fields to become more uniform in their lower parts and on the sides of the supergroups (where the modulation amplitudes get much smaller); (v) there is a set-down in the mean free surface accompanying the peaks of the wave envelope and a smaller set-up accompanying their troughs.

(3.2) The induced flow accompanying an envelope soliton

We consider here the case of infinitely deep water. In this case we assume that the induced mean flow ϕ_o is of order ε^2 ($\phi_{10} = 0(\varepsilon)$; $\phi_{20} = 0(\varepsilon)$). For infinitely deep water, system (3.1) to (3.4) becomes:

$$\phi_{10_{zz}} + \frac{\varepsilon^2}{C_g^2}\phi_{10_{\tau\tau}} = 0 \qquad -\infty < z < 0 \tag{3.2.1}$$

$$\lim_{z\to-\infty} \phi_{10_z} = 0 \tag{3.2.2}$$

$$\phi_{10_z} = k(|\tilde{\psi}|^2)_\tau \quad , \; z = 0 \tag{3.2.3}$$

$$i\tilde{\psi}_\xi - \frac{1}{g}\tilde{\psi}_{\tau\tau} - \varepsilon^{-2}k^3|\tilde{\psi}|^2\,\tilde{\psi} = 0, \qquad z = 0 \tag{3.2.4}$$

Introducing the dimensionless variables

$$\begin{aligned} T &= \frac{\omega\tau}{2} \\ X &= -k\xi \\ Z &= \varepsilon kz \\ \tilde{\Phi} &= \frac{k^2}{\varepsilon\omega}\Phi_{10}, \qquad \psi = \frac{k}{\varepsilon}\tilde{\psi} \end{aligned} \tag{3.2.5}$$

Note that X, T, Z and ψ in the present section are slighly different from their previous definition.

System (3.2.1) to (3.2.4) becomes

$$\tilde{\Phi}_{ZZ} + \tilde{\Phi}_{TT} = 0 \tag{3.2.6}$$

$$\lim_{\tilde{Z}\to-\infty} \tilde{\Phi}_Z = 0 \tag{3.2.7}$$

$$\tilde{\Phi}_Z = \tfrac{1}{2}(|\psi|^2)_T, \qquad \tilde{Z} = 0 \tag{3.2.8}$$

$$i\psi_x + \frac{\psi_{TT}}{4} + |\psi|^2\psi = 0, \qquad Z = 0 \tag{3.2.9}$$

One well-known solution of equation (3.2.9) which decays at $T = \pm\infty$ ($x = \pm\infty$) is the envelope soliton, given by:

$$\psi(X,T) = \mathrm{sech}(\sqrt{2}\,T)\, e^{ix/2} \tag{3.2.10}$$

Substituting (3.2.10) in (3.2.8) yields

$$\tilde{\Phi}_Z = \tfrac{1}{2}\frac{\partial}{\partial T}\{\mathrm{sech}^2(\sqrt{2}\,T)\}, \qquad Z = 0 \tag{3.2.11}$$

Equations (3.2.6), (3.2.7) and (3.2.11) define a Neumann problem in the lower half plane, which was solved utilizing the Fourier transform method, with the following result.

$$\tilde{\Phi} = T \sum_{n=0}^{\infty} (Z-\alpha_n)/[T^2+(Z-\alpha_n)^2]^2 \tag{3.2.12}$$

where

$$\alpha_n = \frac{\pi}{\sqrt{2}}(\tfrac{1}{2}+n) \tag{3.2.13}$$

All technical details are given in appendix 6. Let U and V be the induced mean flow velocity components in the directions x and z respectively. From (3.2.12) and (3.2.5) it follows

$$u = \frac{Uk}{\varepsilon^3\omega} = \sum_{n=0}^{\infty} (Z-\alpha_n) \left[(Z-\alpha_n)^2 - 3T^2\right] / \left[T^2+(Z-\alpha_n)^2\right]^3 \tag{3.2.14}$$

$$v = \frac{Vk}{\varepsilon^3\omega} = T \sum_{n=0}^{\infty} \left[T^2-3(Z-\alpha_n)^2\right] / \left[T^2+(Z-\alpha_n)^2\right]^3 \tag{3.2.15}$$

The dimensionless stream function is given by

$$\tilde{\Psi}(x,z) = \frac{k^2}{\varepsilon^2\omega}\Psi = \tfrac{1}{2} + \tfrac{1}{2} \sum_{n=0}^{\infty} \frac{T^2-(Z-\alpha_n)^2}{\left[T^2+(Z-\alpha_n)^2\right]^2} \tag{3.2.16}$$

Note that $\sum_{n=0}^{\infty} \alpha_n^{-2} = 1$ and that the first term on the r.h.s. of eq. (3.2.16) was chosen to render $\Psi(0,0) = 0$. The streamlines of the induced mean flow field are shown in Fig. 12, the total flux, per unit width, involved in this flow is given by

$$q = \Psi(\infty,0) - \Psi(0,0) = \tfrac{1}{2}\,\varepsilon^2 \frac{\omega}{k^2} \tag{3.2.17}$$

This value is equal to the Stokesian mass transport at the peak of the wave packet, as it should be.

4. CONCLUDING REMARKS

The main result of the present study is the development of an analytical solution which is able to provide detailed information about the physical quantities involved in the shoaling of wave groups.

The quantities include: the variations of the mean free surface (also called set-up and set-down); the induced mean flow; and the supergroups (group of groups) which to our knowledge have not been discussed in the past.

The improvement in our understanding of the above mentioned quantities is of practical importance since it is believed that they are related with such phenomena as surf-beats, longshore cellular structure and Harbour resonance.

Appendix A: Decoupling of eqs. (3.3), (3.4) for periodical boundary conditions

If $\tilde{\psi}$ and ϕ_{10} are periodical of period γ, the solution of (3.1) to (3.7) is given by:

$$\tilde{\psi}(\xi,\tau) = \varepsilon\left(\frac{g^3}{2\omega^5 C_g}\right)^{\frac{1}{2}} e^{-i\overline{a^2}\int_{x_\infty}^{x} \frac{\alpha_2}{C_g} dx} \{D_o(\xi) +$$

$$+ \sum_{n=1}^{\infty} \{D_n(\xi)\, e^{\frac{2\pi i n\tau}{\gamma}} + D_{-n}(\xi)\, e^{\frac{-2\pi i n\tau}{\gamma}}\} \tag{A.1}$$

where α_2 is given by (2.2.14).

$$\phi_{10}(\xi,\tau,z) = Q_1(\xi)\tau + Q_2(\xi) + \sum_{n=1}^{\infty} a_n(z)\{ b_n(\xi)e^{\frac{2\pi i n\tau}{\gamma}} +$$

$$+ b_{-n}(\xi)e^{\frac{-2\pi i n\tau}{\gamma}}\} \tag{A.2}$$

Substituting (A.2) in (3.1), and using the b.c. (3.2) yields

$$a_n = \cosh\left[\frac{\varepsilon n\cdot 2\pi(n+h)}{C_g\gamma}\right] \quad , \quad n = 1,2,... \tag{A.3}$$

(same as (3.1.2))

From (A.3) and the b.c. (3.3) we obtain

$$b_n(\xi) = \frac{\varepsilon^{-2}g^3\beta_2}{2\omega} \frac{C_n(\xi)}{\frac{ng2\pi}{\varepsilon C_g\gamma}\sinh\left[\frac{2\pi\varepsilon nh}{C_g\gamma}\right]-\left[\frac{2\pi n}{\gamma}\right]^2\cosh\left[\frac{2\pi n\varepsilon h}{C_g\gamma}\right]} \tag{A.4}$$

(same as (3.1.3))

where $C_n(\xi)$ are the Fourier coefficients of $|\tilde{\psi}|^2_\tau$:

$$|\tilde{\psi}|^2_\tau = \sum_{n=1}^{\infty} \{C_n(\xi)e^{\frac{2\pi i n\tau}{\gamma}} + C_{-n}(\xi)e^{\frac{-2\pi i n\tau}{\gamma}}\}(C_n(\xi) < 0(\varepsilon^2)) \tag{A.5}$$

(Same as (3.1.4))

Substituting (A.3) and (A.4) in (A.2) and differentiating twice with respect to τ yields:

$$\phi_{10_{\tau\tau}}\Big|_{z=0} = \sum_{n=1}^{\infty} \frac{\varepsilon^{-2}g^2\beta_2}{2\omega} \frac{\{C_n(\xi)e^{\frac{2\pi in\tau}{\gamma}} + C_{-n}(\xi)e^{\frac{-2\pi in\tau}{\gamma}}}{1 - \frac{\gamma g}{2\pi n\varepsilon C_g} \operatorname{th}[\frac{2\pi n\varepsilon h}{C_g\alpha}]} \tag{A.6}$$

Thus from (A.5) and (A.6) follows the relation

$$\phi_{10_{\tau\tau}}\Big|_{z=0} = \Gamma_1|\tilde{\psi}|^2_\tau + \sum_{n=2}^{\infty}(\Gamma_n-\Gamma_1)\{C_n(\xi)e^{\frac{2\pi in\tau}{\gamma}} + C_{-n}e^{\frac{-2\pi in\tau}{\gamma}}\} \tag{A.7}$$

where

$$\Gamma_1 = \frac{\varepsilon^{-2}g^2\beta_2}{2\omega\ (1 - \frac{\gamma g}{2\pi\varepsilon C_g}\ \operatorname{th}\ [\frac{\varepsilon h 2\pi}{C_g\gamma}])} \tag{A.8}$$

$$\Gamma_n = \frac{\varepsilon^{-2}g^2\beta_2}{2\omega(1- \frac{\gamma g}{2\pi n\varepsilon C_g}\ \operatorname{th}[\frac{2\pi n\varepsilon h}{C_g\gamma}])}, \quad n = 2,3,\ldots \tag{A.9}$$

It can be easily seen that $\Gamma_n-\Gamma_1 < 0(\varepsilon^{-2})$ for all n. On the other hand $C_n(\xi)$ are Fourier coefficients. Thus there exists a N, such that

$$\sum_{n=N}^{\infty} \{C_n(\xi)\ e^{\frac{2\pi in\tau}{\gamma}} + C_{-n}(\xi)e^{\frac{-2\pi in\tau}{\gamma}}\} < \varepsilon^3$$

and then

$$\sum_{n=N}^{\infty} (\Gamma_n-\Gamma_1)\ \{\ C_n e^{\frac{2\pi in\tau}{\gamma}} + C_{-n}e^{\frac{-2\pi in\tau}{\gamma}}\} < 0(\varepsilon) \tag{A.10}$$

For $n < N$:

$$\Gamma_n - \Gamma_1 < 0(\varepsilon^{-1}) \tag{A.11}$$

From (A.7), (A.10) and (A.11) we finally obtain the relation

$$\phi_{10_{\tau\tau}}\Big|_{z=0} = \Gamma_1 |\tilde{\psi}|^2_\tau + 0(\varepsilon) \tag{A.12}$$

And using the relation

$$\Gamma_1 = \frac{\varepsilon^{-2} g^2 \beta_2}{2\omega(1 - \frac{gh}{C_g^2})} + 0(\varepsilon^{-1}) \tag{A.13}$$

which can be easily verified, we obtain

$$\phi_{10_{\tau\tau}}\Big|_{z=0} = \Gamma |\tilde{\psi}|^2_\tau + 0(\varepsilon) \tag{A.14}$$

where

$$\Gamma = \frac{\varepsilon^{-2} g^2 \beta_2}{2\omega\ (1 - \frac{gh}{C_g^2})} \tag{A.15}$$

Equation (A.14) is identical to (2.2.4) ($A = \frac{g}{2\omega\varepsilon} \tilde{\psi}$), which was obtained for the case of shallow water. Integrating (A.14) with respect to τ yields

$$\phi_{10_{\tau}}\Big|_{z=0} = \Gamma |\tilde{\psi}|^2 + Q(\xi) + 0(\varepsilon) \tag{A.16}$$

From here we proceed in a similar way as in section (2.2) and obtain that $Q(\xi)$ is given by (2.2.8), ϕ_{20} by (2.2.11), and $Q_1(\xi) = \phi_{20}(\xi)$.

Finally substituting ϕ_{10_τ}, $Q(\xi)$ and ϕ_{20} in (3.4) we obtain that the wave envelope $\tilde{\psi}(\xi,\zeta)$ is governed by the nonlinear Schrödinger equation (2.2.12).

$$\frac{i}{2C_g}\frac{\partial C_g}{\partial \xi}\tilde{\psi} + i\tilde{\psi}_\xi + \frac{C''_g}{2C_g^3}\tilde{\psi}_{\tau\tau} - \frac{\varepsilon^{-2}\alpha_1}{C_g}|\tilde{\psi}|^2\tilde{\psi} = \frac{\varepsilon^{-2}\alpha_2}{C_g}\overline{|\tilde{\psi}|^2}\tilde{\psi}$$

where α_1 and α_2 are given by (2.2.13) and (2.2.14) respectively.

For deep water ($\varepsilon h >> \gamma C_g$), eqs. (A.8) and (A.9) become

$$\Gamma_1 = -\frac{\varepsilon^{-1}g^2\beta_2\pi C_g}{\omega\gamma g} \tag{A.17}$$

and

$$\Gamma_n = -\frac{\varepsilon^{-1}g^2\beta_2\pi n C_g}{\omega\gamma g} \tag{A.18}$$

respectively, and one can easily verify that $\phi_{10}\big|_{z=0} = O(\varepsilon)$.

Appendix B - Solution of Equations (3.1a.3) and (3.1a.4) for constant depth

Input data: $J_1, J_3\ P$; $I = J_3 - J_1^2$

$$\tilde{c}_1 = \frac{1}{7}(4J_1 - P)\{1 - [1 - \frac{14I}{(4J_1-P)^2}]^{\frac{1}{2}}\} \quad (B.1)$$

$$\tilde{c}_2 = P\,[1 - (1 + \frac{2I}{P^2})^{\frac{1}{2}}] \quad (B.2)$$

$$c = \max(\tilde{c}_1, \tilde{c}_2); \qquad d = \min(c_1, c_2) \quad (B.3a,b)$$

$$e = 2P; \qquad b = \frac{2}{7}(4-P) \quad (B.3c,d)$$

$$r^2 = 1 - \frac{e-b}{e} \cdot \frac{c-d}{b} \quad (B.4)$$

$$y = -\frac{2\pi^2}{P}\sqrt{7eb} \cdot X \quad (B.5)$$

cd(y,r) is a Jacobian elliptic function of the argument y with modulus r.

$$\tilde{z} = \frac{eb(1-cd^2)}{e-b \cdot cd^2} \quad (B.6)$$

$$|D_1| = \sqrt{\tilde{z}/2}; \qquad |D_o| = \sqrt{J_1 - \tilde{z}} \quad (B.7a,b)$$

$$\cos[2(\arg D_1 - \arg D_o)] = \frac{I - 1.5\tilde{z}^2 + (P-2J_1) \cdot \tilde{z}}{2\tilde{z}(J_1 - \tilde{z})} \quad (B.7c,d)$$

Appendix C - Derivation of the Equation for $J_3(X)$

Let us suppose that μ is a given function of X. Differentiating equation (3.1.8) with respect to T and multiplying by $\psi_T{}^*$ we obtain

$$i\psi_T^* \frac{\partial\psi_T}{\partial X} + \psi_T^* \frac{\partial^2}{\partial T^2} \psi_T + \mu(X)\cdot|\psi|^2|\psi_T|^2 + \mu(X)\cdot\psi\psi_T{}^*|\psi|^2{}_T=0 \qquad (C.1)$$

Subtracting from (C.1) its complex conjugate and dividing the result by $\mu(X)$ yields

$$\frac{i}{\mu(X)} \cdot \frac{\partial|\psi_T|^2}{\partial X} + \frac{1}{\mu(X)} \frac{\partial}{\partial T} (\psi_T{}^*\psi_{TT}-\psi_T\psi_{TT}^*)+ |\psi|^2{}_T(\psi\psi_T^*-\psi^*\psi_T) = 0 \qquad (C.2)$$

Multiplying equation (3.1.8) by ψ^*, taking the complex conjugate and substracting it from the resulting expression, we obtain

$$i \frac{\partial}{\partial X} |\psi|^2+ \frac{\partial}{\partial T} (\psi_T\psi^*-\psi_T^*\psi) = 0 \qquad (C.3)$$

Multiplying (C.3) by $|\psi|^2$ yields

$$\frac{i}{2} \frac{\partial|\psi|^4}{\partial X} + |\psi|^2\frac{\partial}{\partial T} (\psi_T\psi^*-\psi_T^*\psi) = 0 \qquad (C.4)$$

Subtracting (C.4) from (C.2) gives

$$\frac{i}{\mu(X)} \cdot \frac{\partial}{\partial X} |\psi_T|^2- \frac{i}{2} \frac{\partial|\psi|^4}{\partial X} + \frac{\partial}{\partial T} \{\frac{(\psi_T^*\psi_{TT}-\psi_T\psi_{TT}^*)}{\mu(X)} +|\psi|^2(\psi_T^*\psi-\psi_T\psi^*)\}=0 \qquad (C.5)$$

which can be written as

$$i \frac{\partial}{\partial X} \{ \frac{|\psi_T|^2}{\mu(X)} - \tfrac{1}{2}|\psi|^4\} + \frac{i|\psi_T|^2\mu'}{\mu^2} + \frac{\partial}{\partial T} \{\frac{(\psi_T^*\psi_{TT}-\psi_T\psi_{TT}^*)}{\mu} +$$

$$+ |\psi|^2(\psi_T^*\psi-\psi_T\psi^*)\} = 0 \qquad (C.6)$$

Integrating (C.6) with respect to T from 0 to 1, and using the fact that ψ is periodic, with period 1 we obtain:

$$\frac{\partial}{\partial X}\left\{\int_0^1\left[\frac{|\psi_T|^2}{\mu(X)} - \frac{|\psi|^4}{2}\right]dT\right\} + \frac{\mu'}{\mu^2}\int_0^1 |\psi_T|^2 dT = 0 \tag{C.7}$$

which can be written as:

$$\frac{\partial}{\partial X} J_3(X) = \frac{2\mu'}{\mu^2}\int_0^1 |\psi_T|^2 dT \tag{C.8}$$

where $J_3(X) = \int_0^1 (|\psi|^4 - \frac{2}{\mu(X)}|\psi_T|^2)dT$.

Substituting the approximate solution (3.1a.2) into the right hand side of (C.8) we obtain the following approximate equation for J_3

$$\frac{\partial J_3(X)}{\partial X} = 8\pi^2 \frac{\mu'}{\mu^2}\tilde{z}(X) \tag{C.9}$$

where $\tilde{z}(X) = 2|D_1|^2$.

Integrating eq. (C.9) yields

$$J_3(X) = J_3(0) + 8\pi^2\int_0^X \frac{\mu'}{\mu^2}\tilde{z}(X)dX = J_3(0) + 8\pi^2\int_{\mu_o}^{\mu}\frac{\tilde{z}(\mu)}{\mu^2}d\mu \tag{C.10}$$

and using (3.1.17) one finally obtains

$$J_3(P(X)) = J_3(P_o) - \int_{P_o}^{P}\tilde{z}(P)dP \tag{C.11}$$

Equation (C.11) is an integral equation for $J_3(P)$. If we suppose that $J_3(P(X))$ varies in such a way that $J_1^2 - J_3 << 1$, then $\tilde{z}(P)$ may be replaced by the solution for constant depth (B.6) so that

$$J_3(P) = J_3(P_o) - \int_{P_o}^{P} \frac{2P(4-P)(1-cd^2)}{7P-(4-P)cd^2}\,dP \tag{C.12}$$

where cd is the Jacobian ellyptic function with argument

$$y = -4\pi^2\left[\frac{(4-P)}{P}\right]^{\frac{1}{2}} X \tag{C.13}$$

and modulus r given by

$$r^2 = 1 - \frac{(e-b)(c-d)}{eb} = 1 - 4\,\frac{(4P-2)}{P^2(4-P)^2}\,(J_3-J_1^2) + 0(J_3-J_1^2)^2 \tag{C.14}$$

e,b,c, and d are given in Appendix B.

After some algebra, and using the relations

$$cd = \frac{cn}{dn}$$

$$cn^2 = 1-sn^2 \tag{C.15}$$

$$dn^2 = 1-r^2sn^2$$

$$r'^2 = 1-r^2$$

where cn, dn and sn are Jacobian ellyptic functions, equation (C.12) is written in the form

$$J_3(P) = J_3(P_o) - \int_{P_o}^{P} \left(\frac{eb\cdot r'^2}{e-b} \quad \frac{sn^2}{1-q^2sn^2}\right)dP \tag{C.16}$$

where

$$q^2 = 1 - \frac{(c-d)}{b} \tag{C.17}$$

The integrand in (C.16) is an oscillatory function of the argument y, and if we again apply the very mild topography assumption, it is justified to replace it by its average as follows

$$\left(\frac{eb^2r'^2}{e-b} \cdot \frac{sn^2}{1-q^2sn^2}\right) = \frac{1}{K}\int_{o}^{K} \frac{(r')^2eb}{e-b}\,\frac{sn^2}{1-q^2sn^2}\,dy =$$

$$= \frac{(r')^2eb}{(e-b)K}\int_{o}^{K} \frac{sn^2(y)\,dy}{1-q^2sn^2(y)} = \frac{(r')^2eb}{e-b}\;\frac{Z_J(p,r)}{\sqrt{q^2(1-q^2)r^2-q^2)}} \tag{C.18}$$

where $4K(P)$ is the period of the sn Jacobian ellyptic function, $Z_J(p,r)$ is the Jacobian Zeta function and

$$p = \sin^{-1}(\frac{q}{r}) \tag{C.19}$$

In order to obtain (C.18), equation (414.02) from Byrd and Friedman (1971) was used.

Substituting (C.18), into (C.16) one obtains, after some algebra

$$J_3(P) \simeq J_3(P_o) - \int_{P_o}^{P} \sqrt{eb}\; J_3(p,r)dP \tag{C.20}$$

or

$$\frac{\partial J_3}{\partial P} \simeq -\sqrt{eb}\;\; Z_J(p,r) \tag{C.21}$$

where

$$\sin^2 p \simeq 1 - \frac{1}{e}\delta + 0(\delta^2) \tag{C.22}$$

$$r^2 = \{1 - \frac{(e-b)}{eb}\,\delta\} \tag{C.23}$$

$$\delta = (c-d) = \frac{4}{P(4-P)}\,(J_3 - J_1^2) << 1 \tag{C.24}$$

Finally, using the fact that $\sin^2 p = 1+0(\delta)$, and $r^2=1+0(\delta)$, we replace $Z_J(p,r)$ in (C.21) by a simpler expression as follows.

The function $Z_J(p,r)$ is defined by

$$Z_J(p,r) = E(p,r) - \frac{E}{K}\cdot F(p,r) \tag{C.25}$$

Here E, K, $E(p,r)$ and $F(p,r)$ are the Ellyptic Integrals. (See Byrd and Friedman (1971)).

Using the following limiting values

$$\lim_{\sin\beta\to 1} E(p,k) = E(k) \tag{C.26}$$

$$\lim_{r^2\to 1} E(r) = 1 \tag{C.27}$$

the function $Z_J(p,r)$ is approximated by

$$Z_J(p,r) \simeq 1 - \frac{F(p,r)}{K} = \frac{K-F(p,r)}{K} \stackrel{\Delta}{=} \frac{F(\pi/2,r)-F(p,r)}{K} = \frac{F(\tilde{\varphi},r)}{K} \tag{C.28}$$

where

$$\tilde{\varphi} = \cos^{-1}\left\{ \frac{\sin p\,\sqrt{(1-r^2)(1-r^2\sin^2 p)}}{1-r^2\sin^2 p}\right\} \simeq \cos^{-1}\sqrt{\frac{1-r^2}{1-r^2\sin^2 p}} \tag{C.29}$$

see Byrd and Friedman (1971) Eq. (116.01).

From (C.22) and (C.23) we obtain

$$\tilde{\varphi} = \cos^{-1}\sqrt{\frac{e-b}{e}} \tag{C.30}$$

UsingEq.(902.01)on page 300, in Byrd and Friedman (1971) yields

$$F(\tilde{\varphi},r) \simeq \ln\left(\frac{1+\sin\tilde{\varphi}}{\cos\tilde{\varphi}}\right) + O(r'^2) = \tfrac{1}{2}\ln\left(\frac{1+\sqrt{\frac{b}{e}}}{1-\sqrt{\frac{b}{e}}}\right) + O(r'^2) \tag{C.31}$$

Finally, replacing K by its approximate value

$$K \simeq \ln\left(\frac{4}{r'}\right), \qquad r \simeq 1 \tag{C.32}$$

(see 112.01 page 11, Byrd and Friedman (1971), we obtain from (C.28), (C.31) and (C.32) the following equation for $J_3(P)$:

$$\frac{J_3(P)}{P} = -\frac{\sqrt{eb}}{2}\cdot\frac{\ln\left(\frac{1+\sqrt{\frac{b}{e}}}{1-\sqrt{\frac{b}{e}}}\right)}{\ln\left|\frac{4}{\sqrt{\frac{(e-b)(c-d)}{eb}}}\right|} \tag{C.33}$$

The above equation is written, using (B.3) as

$$\frac{\partial J_3(P)}{\partial P} = - 2\sqrt{\frac{P}{7}(4-P)} \cdot \frac{\ln\left(\dfrac{1+\sqrt{\frac{(4-P)}{7P}}}{1-\sqrt{\frac{(4-P)}{7P}}}\right)}{\ln\left|\dfrac{2P^2(4-P)^2}{(2P-1)(J_3(P)-J_1^2)}\right|} \tag{C.34}$$

Finally, introducing

$$I(P) = J_3(P) - J_1^{\,2} \tag{C.35}$$

$$\frac{\partial I(P)}{\partial P} = - \sqrt{\frac{P}{7}(4-P)} \cdot \frac{2\ln\left(\dfrac{1+\sqrt{\frac{(4-P)}{7P}}}{1-\sqrt{\frac{(4-P)}{7P}}}\right)}{\ln\left|\dfrac{2P^2(4-P)^2}{(2P-1)I(P)}\right|} \tag{C.36}$$

Appendix D - Solution of the Neumann problem (3.2.6), (3.2.7), (3.2.11)

We consider the problem

$$\tilde{\Phi}_{ZZ} + \tilde{\Phi}_{TT} = 0 \tag{3.2.6}$$

$$\lim_{Z \to -\infty} \tilde{\Phi}_Z = 0 \tag{3.2.7}$$

$$\tilde{\Phi}_Z = \tfrac{1}{2} \frac{\partial}{\partial T} \{\mathrm{sech}^2 (\sqrt{2}\, T)\}, \qquad z = 0 \tag{3.2.11}$$

Equation (3.2.6) with the boundary conditions (3.2.7) and (3.2.11) constitute a Neumann problem, which is solved for the induced flow.

Substituting the following Fourier Transform of the induced flow potential function

$$f(\lambda,Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\lambda T} \tilde{\Phi}(X,T,Z)\, dT \tag{D.1}$$

Into (3.2.6), (3.2.7) and (3.2.11) yields

$$f_{ZZ}(\lambda, Z) - \lambda^2 f(\lambda,Z) = 0 \qquad -\infty < Z < 0 \tag{D.2}$$

$$f_Z(\lambda,0) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\lambda T} \frac{\partial}{\partial T}[\mathrm{sech}^2 \sqrt{2} T]\, dT, \qquad Z=0 \tag{D.3}$$

$$\lim_{Z \to -\infty} f_Z(\lambda,Z) = 0 \tag{D.4}$$

The solution of (D.2), (D.3), (D.4) is given by

$$f_Z(\lambda,Z) = \frac{1}{2\sqrt{2\pi}} e^{|\lambda| Z} \int_{-\infty}^{+\infty} e^{i\lambda\tau} \frac{\partial}{\partial \tau} [\mathrm{sech}^2 \sqrt{2}\tau]\, d\tau \tag{D.5}$$

Then, by means of the inverse Fourier Transform, we obtain

$$\tilde{\Phi}_Z(X,T,Z) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} e^{-i\lambda T}\, e^{|\lambda|Z}\int_{-\infty}^{+\infty} e^{i\lambda\tau}\,\frac{\partial}{\partial\tau}\,[\mathrm{sech}^2\sqrt{2}\tau]\,d\tau d\lambda =$$

$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{\partial}{\partial\tau}\,(\mathrm{sech}^2\sqrt{2}\tau)\Big[\int_{-\infty}^{+\infty} e^{i\lambda(\tau-T)+|\lambda|Z}d\lambda\Big]\,d\tau \qquad (D.6)$$

The integral in the square brackets can be evaluated, as a sum of two integrals, from $-\infty$ to 0, and from 0 to ∞, yielding the result

$$\tilde{\Phi}_Z(X,T,Z) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{Z}{(\tau-T)^2+Z^2}\,\frac{\partial}{\partial\tau}[\mathrm{sech}^2\sqrt{2}\tau]\;d\tau \qquad (D.7)$$

Finally, integration with respect to Z gives

$$\tilde{\Phi}(X,T,Z) = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\ln[(\tau-T)^2+Z^2]\cdot\frac{\partial}{\partial\tau}[\mathrm{sech}^2\sqrt{2}\tau]d\tau + Q(X,T) \qquad (D.8)$$

From equation (3.2.6) it follows that

$$Q(X,T) = a(X)T + W(X) \qquad (D.9)$$

and imposing the condition

$$\lim_{T\to\pm\infty}\tilde{\phi}_T = 0 \qquad (D.10)$$

yields that Q is only function of X.

$$Q = W(X) \qquad (D.11)$$

On the other hand, replacing τ (D.8) by the new variable $\tau-T$, and integrating twice by parts, yields the result

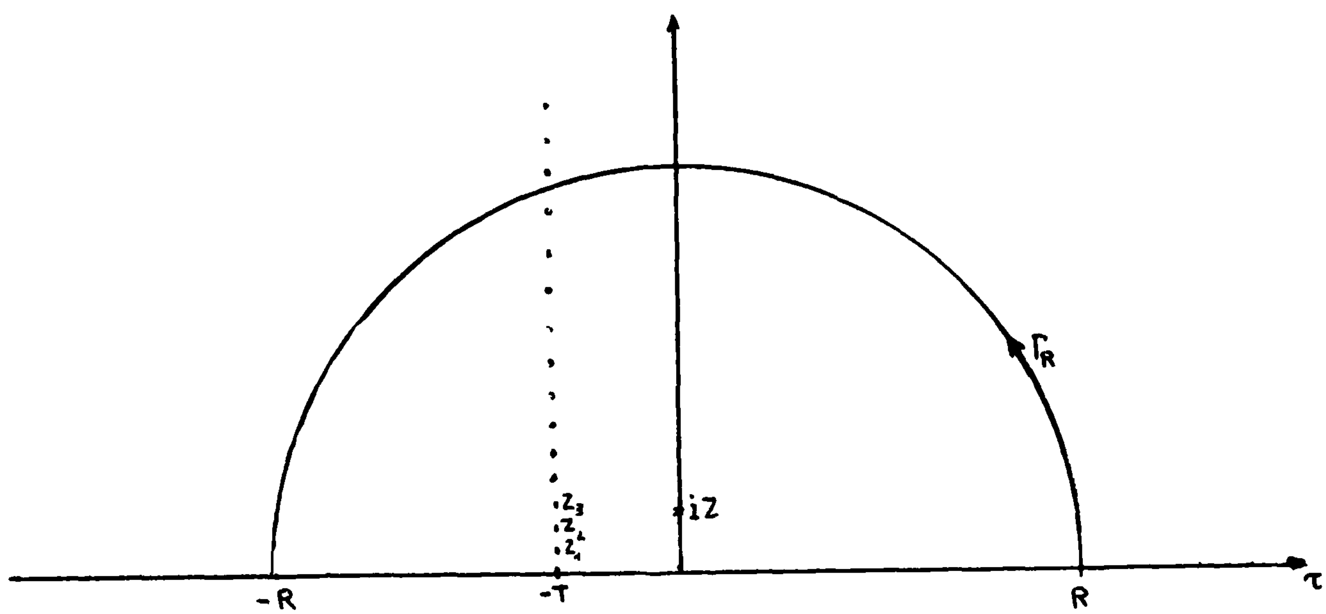
$$\tilde{\phi}(X,T,Z) = \frac{\sqrt{2}}{4\pi}\int_{-\infty}^{+\infty}\frac{(\tau^2-Z^2)}{[\tau^2+Z^2]^2}\ \mathrm{tgh}[\sqrt{2}(\tau+T)]\ d\tau \tag{D.12}$$

The integral in (D.12) is evaluated by considering the complex variable $\eta=\tau+i\zeta$ and the path Γ_R given in figure below. The integrand has the following infinite number of poles in the upper half plane.

$$\eta_o = iZ(\text{Double}) \tag{D.13}$$

$$Z_j = -T + \frac{i\pi}{\sqrt{2}}(\tfrac{1}{2}+j), \qquad j=0,1,2,... \tag{D.13}$$

The path Γ_R is taken between two poles, as shown in the figure



The integral along the half circle tends to zero when $R \to \infty$, and after some algebra we obtain the result:

$$\tilde{\Phi}(X,T,Z) = \frac{i\varepsilon^2}{2}\{\operatorname{sech}^2[\sqrt{2}(T-iZ)] + \frac{1}{4}\cdot\sum_{j=0}^{\infty}\frac{Z_j^2 - Z^2}{(Z_j^2+Z^2)^2} + W(X) \qquad (D.14)$$

where Z_j is given by (D.13)

From the identity

$$\operatorname{sech}^2\zeta = -\sum_{k=-\infty}^{+\infty}\frac{1}{[\zeta+ i\pi\,(\tfrac{1}{2}+k)]^2} \qquad (D.15)$$

(See eq. (3.64) in Carrier et al. (1966)), we obtain the final result

$$\tilde{\phi}(X,T,Z) = \sum_{k=0}^{\infty}\frac{\varepsilon^2 T\{Z-\frac{\pi}{\sqrt{2}}\,(\tfrac{1}{2}+k)\}}{[T^2 + \{Z-\frac{\pi}{\sqrt{2}}\,(\tfrac{1}{2}+k)\}^2]^2} + W(X) \qquad (D.16)$$

REFERENCES

Byrd, P.F. and Friedman, M.D. 1971. Handbook of Ellyptic Integrals for Engineers and Scientists. Springer

Carrier, G.F., Krook, M., and Pearson, C.E. Functions of a complex variable, Theory and Technique. McGraw-Hill.

Davey, A. and Stewartson, K. 1974. On three-dimensional packets of surface waves. Proc. R. Soc. A338, 101-110.

Djordjevic', V.D. and Redekopp, L.G. 1978. On the development of packets of surface gravity waves moving over an uneven bottom. J. Appl. Maths. Phys. 29, 950-962.

Dysthe, K.B., 1980, Note on a modification of the nonlinear Schrödinger equation for application to deep water waves, 1974. Proc. R. Soc. A369-105-114.

Freilich, M.H. and Guza R.T. 1984. Nonlinear effects on shoaling surface gravity waves.Phil. Trans. R. Soc. Lond. A311, 1-41.

Hearn, A.C. 1975. REDUCE 2 User's Manual, University of Utah.

Longuet-Higgins, M.S. 1984. Statistical properties of wave groups in a random sea state. Phil. Trans. R. Soc. Lond. A312, 219-250.

Peregrine, D.H., 1983, Water waves, nonlinear Schrödinger equations and their solutions, J. Austral. Math. Soc. B25, 16-43.

Stiassnie, M. and Peregrine, D.H., 1980. Shoaling of finite-amplitude surface waves on water of slowly-varying depth. J. Fluid Mech. 97, 783-805.

Stiassnie, M. and Kroszynski,U.I., 1982. Long time evolution of an unstable water wave train. J. Fluid. Mech. 116, 207-225.

Stiassnie, M. and Shemer L. 1984. On modifications of the Zakharov equation for surface gravity waves,J. Fluid Mech. 143, 46-67.

Turpin, F.M. Benmoussa, C., and C.C.Mei 1983, Effects of slowly varying depth and current on the evolution of a Stokes wave packet, J. Fluid Mech. 132, 1-23.

Whitham, G.B. 1974. Linear and nonlinear waves. Wiley-Interscience, New York.

List of Figures

1 $I=I(P)$, --- Asymptotic solution, ——— Numerical solution of Eqs. (3.1a.3), (3.1a.4).

2 Exterior-group-envelope $|\psi(o,X)|$ and Interior-group envelope $|\psi(\frac{1}{2},X)|$ for the input conditions $\alpha = 0$, $\beta = 0.158$, (I_o=0.1) --- Asymptotic solution, ——— Numerical solution of Eqs. (3.1a.3), (3.1a.4)

3 Exterior-group-envelope $|\psi(0,X)|$ and Interior-group-envelope $|\psi(\frac{1}{2},X)|$ for the input conditions $\alpha = 0$, $\beta = 0.1$, (I_o=0.04), ——— Numerical solution of O.D.E. (3.1a.3),(3.1a.4) -.-. Numerical solution of N.L.S. (3.1.8)

4 $P=P(K_\infty h)$ for $P_o = 2$.

5 $I_o = I_o(\alpha,\beta)$ for P_o=2 (on $\beta^2 = 0$: I_o=0).

6 The 'supergroup' length ℓ_3 as a function of depth h.

7 The group envelope (---) and wave envelope (——) at t = constant for P_o=2, I_o=0.1, $\varepsilon = 0.2$ and $K_\infty h$ = (a)∞ , (b) 11.2, (c) 5.7 and (d) 4.2.

8 The group-envelopes, for P_o=2, I_o=0.1, at $K_\infty h \simeq$ (a)∞, (b)7.11-4.8, (c)1.32-1.29, (d)1.12-1.11, (e)0.96-0.95.

9 The flow field for P_o=2, I_o=0.1, ε= 0.2 at $K_\infty h = \infty$. (a) group-envelopes and wave envelope (b) mean flow stream-lines, (c) mean free-surfaces.

10 As in 9 for $K_\infty h$ = 11.2

11 As in 9 for $K_\infty h$ = 4.2

12 The wave-induced mean flow field under a deep water envelope soliton

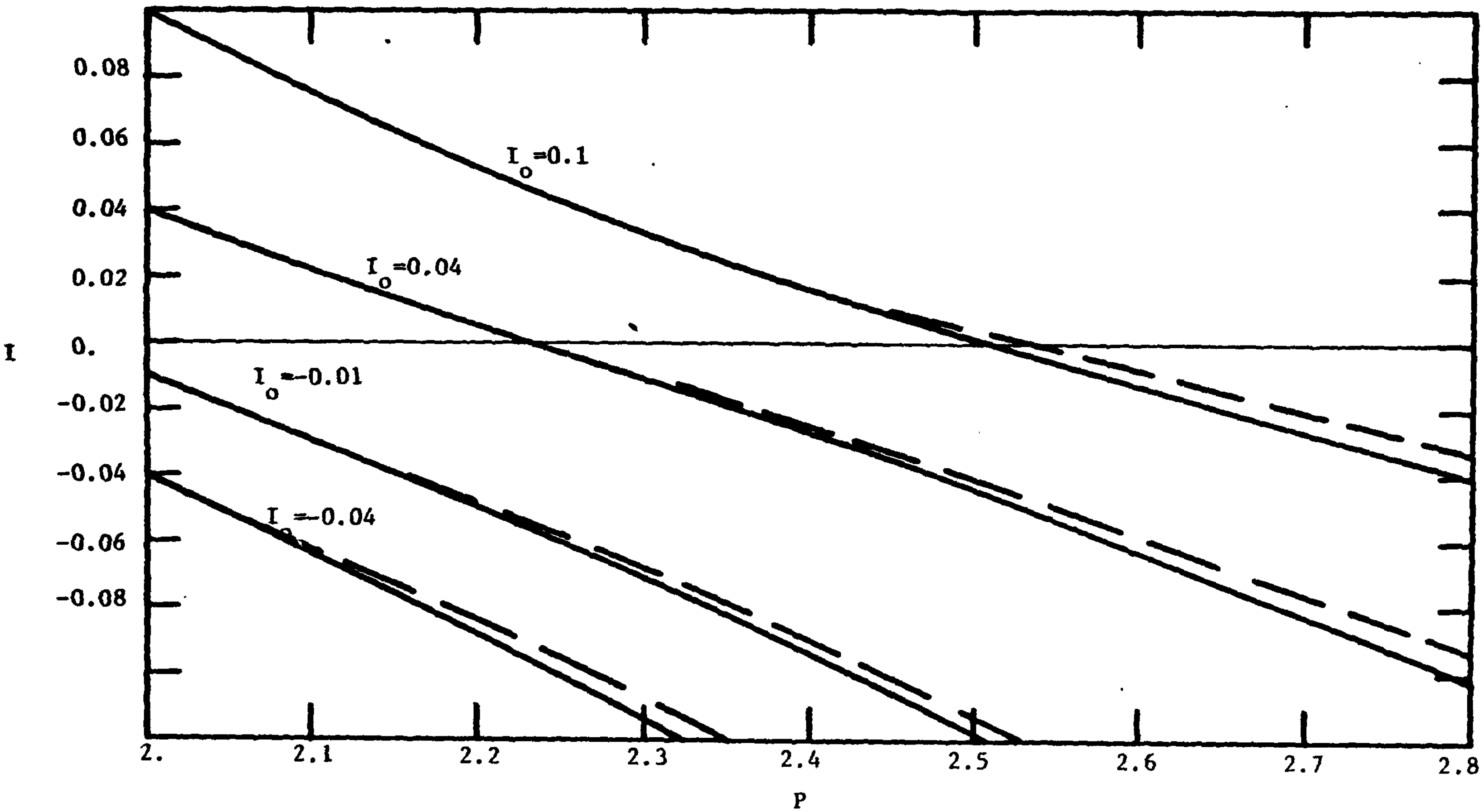


Figure 1

I = I(P), --- Asymptotic solution, —— Numerical solution of Eqs. (3.1a.3),(3.1a.4)

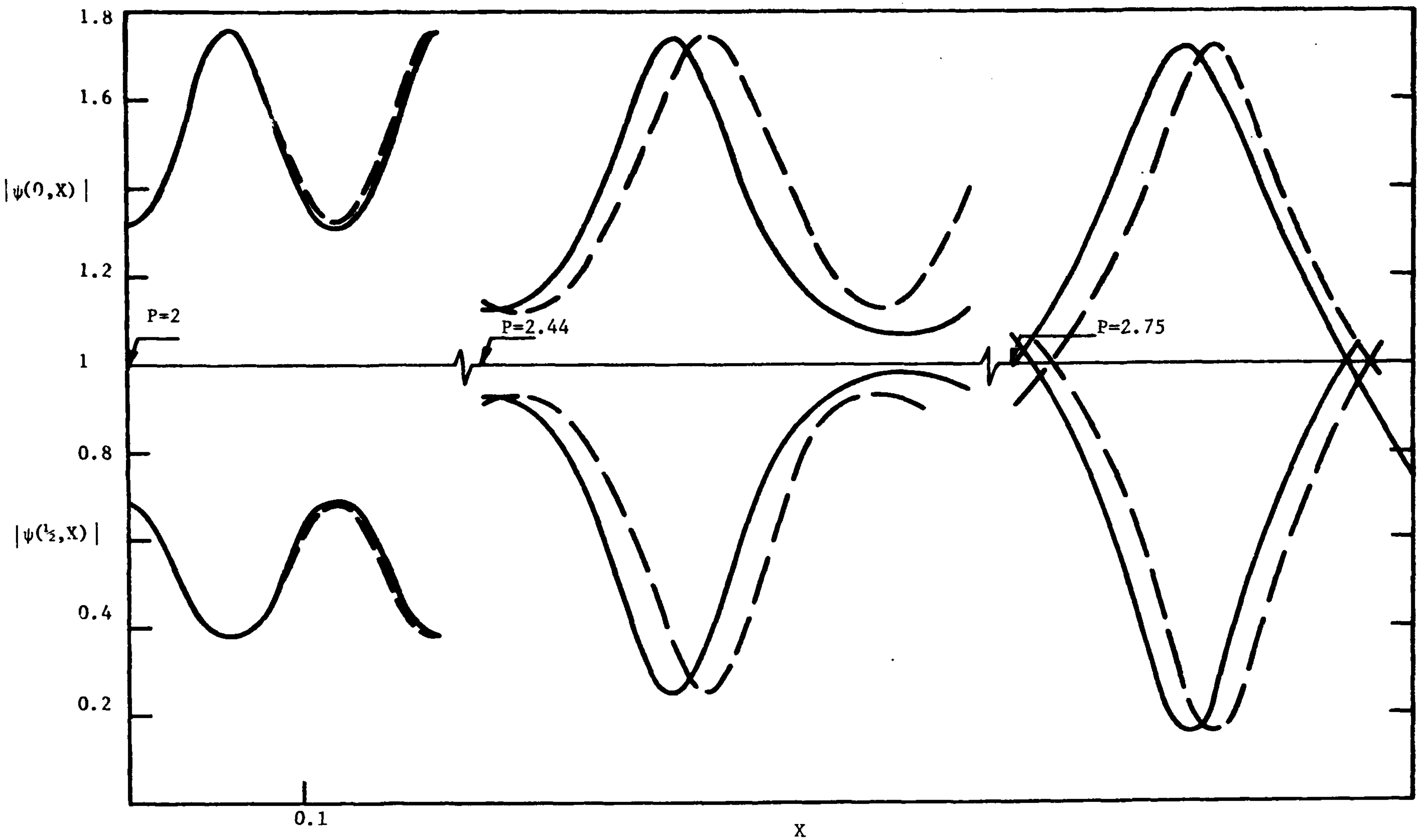


Figure 2

Exterior - group-envelope $|\psi(o,x)|$ and Interior-group envelope $|\psi(\frac{1}{2},x)|$ for the input conditions $\alpha = 0$, $\beta = 0.158$, $(I_o = 0.1)$ ----- Asymptotic solution, —— Numerical solution of Eqs. (3.1a.3),(3.1a.4)

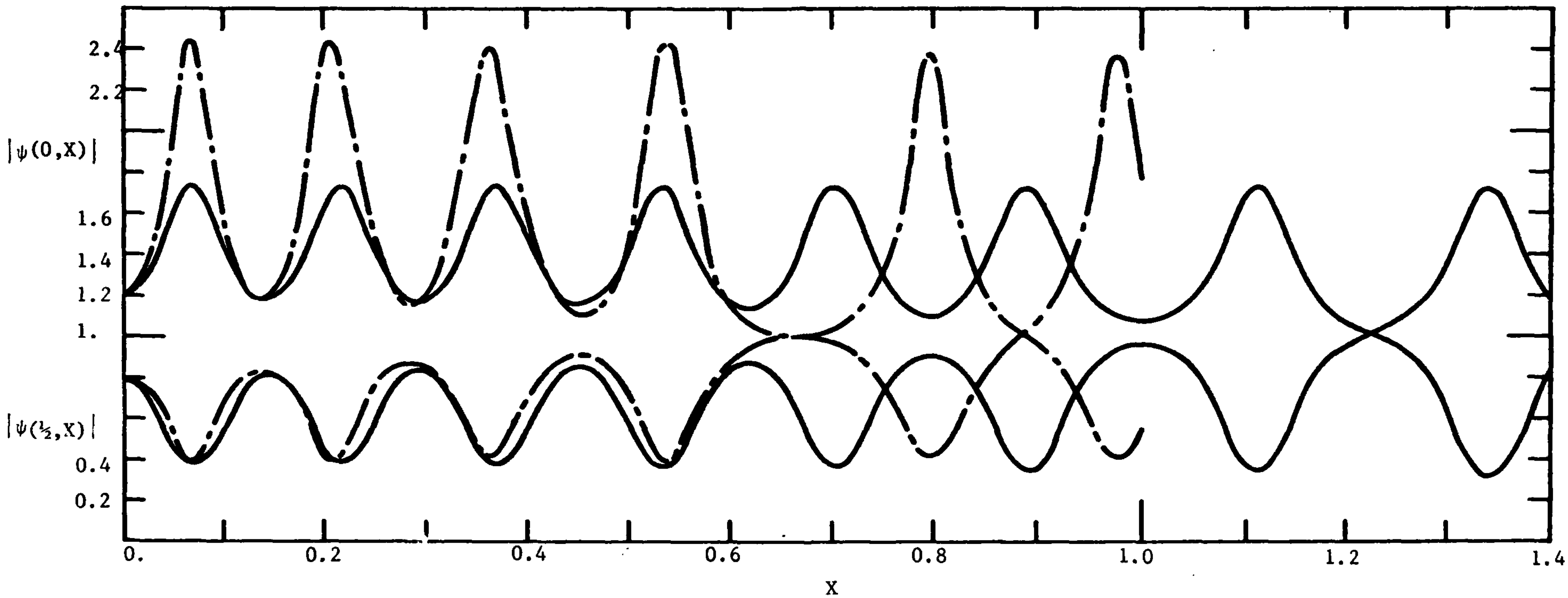
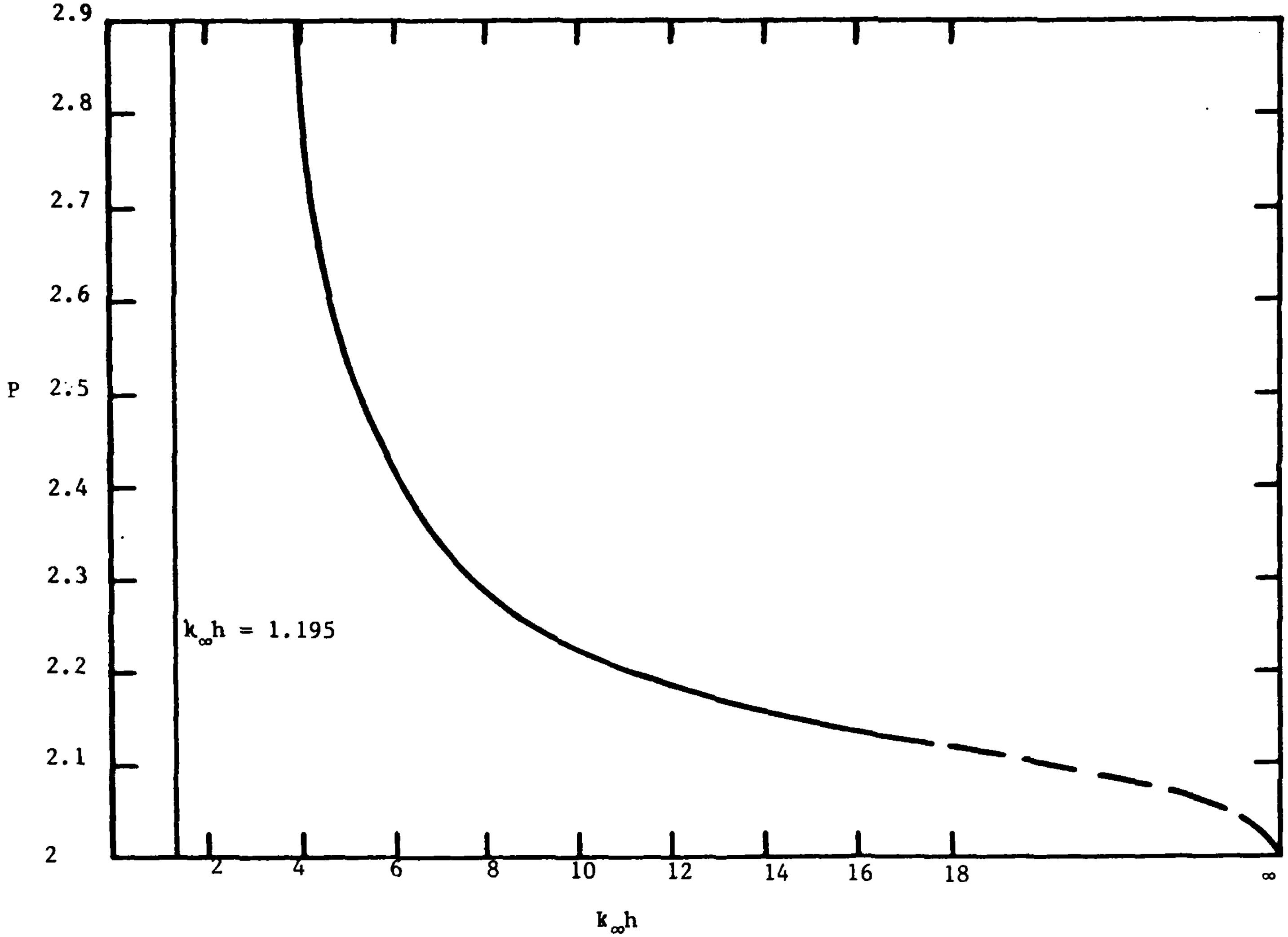


Figure 3

Exterior-group-envelope $|\psi(o,x)|$ and Interior-group-envelope $|\psi(\frac{1}{2},x)|$ for the input conditions $\alpha=0$, $\beta=0.1$, $(I_o=0.04)$, ——— Numerical solution of O.D.E. (3.1a.3), (3.1a.4), -·-·- Numerical solution of N.L.S (3.1.8)



<u>Figure 4</u>

$P = P(k_\infty, h)$ for $P_o = 2$

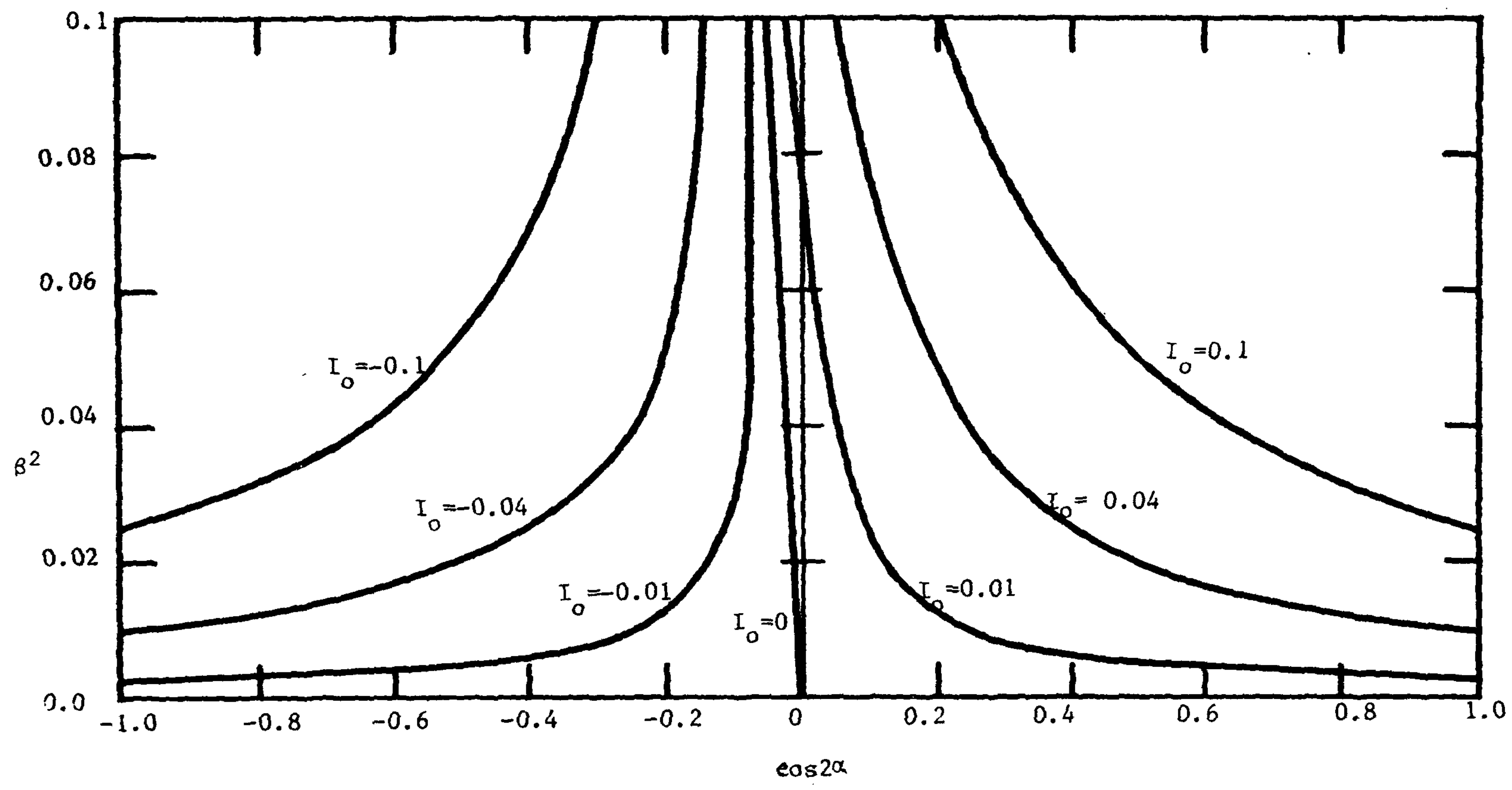


Figure 5

$I_o = I_o(\alpha,\beta)$ for P_o=2 (on β^2= 0:I_o=0)

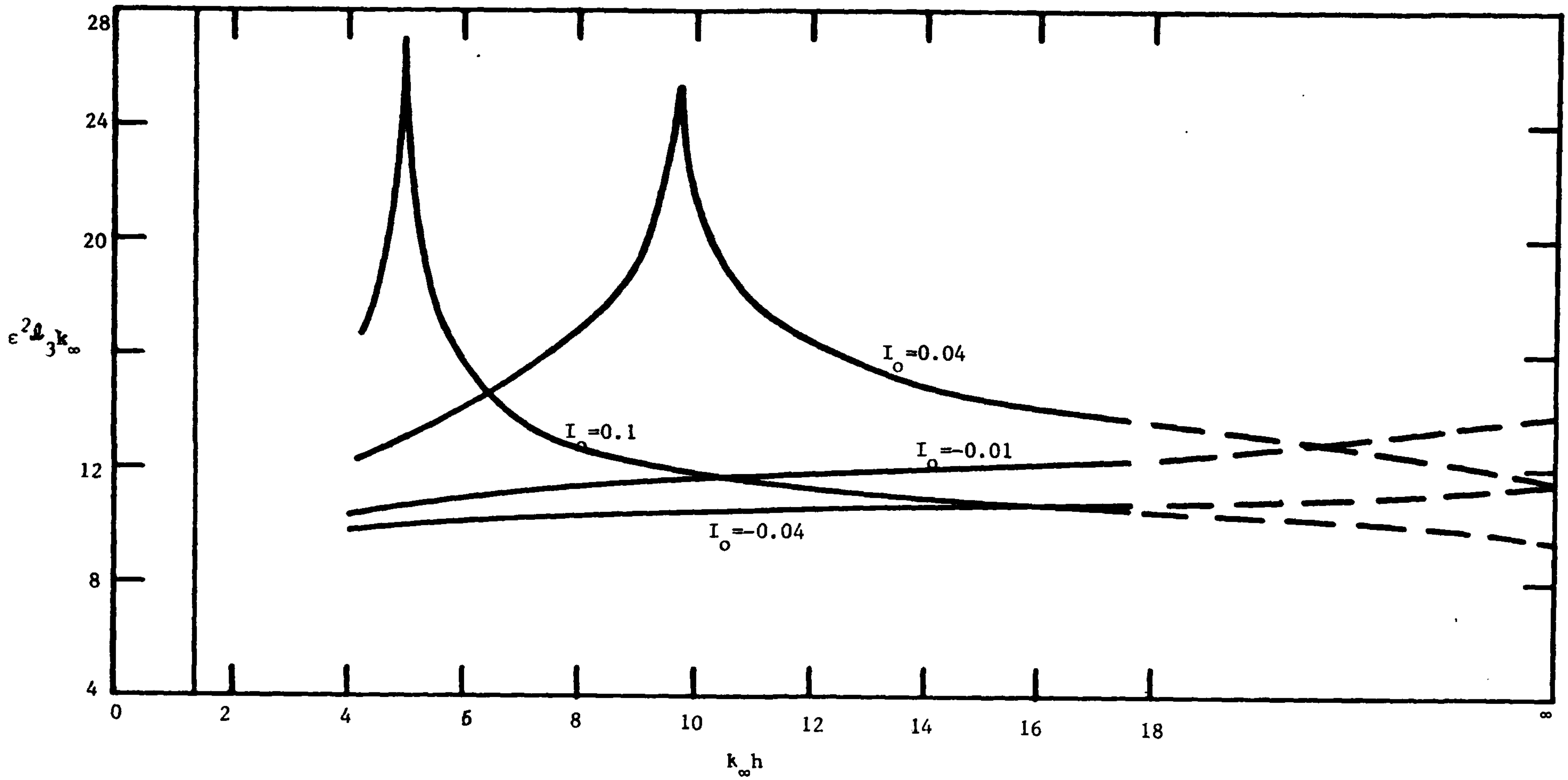


Figure 6

The 'supergroup' length ℓ_3 as a function of depth h.

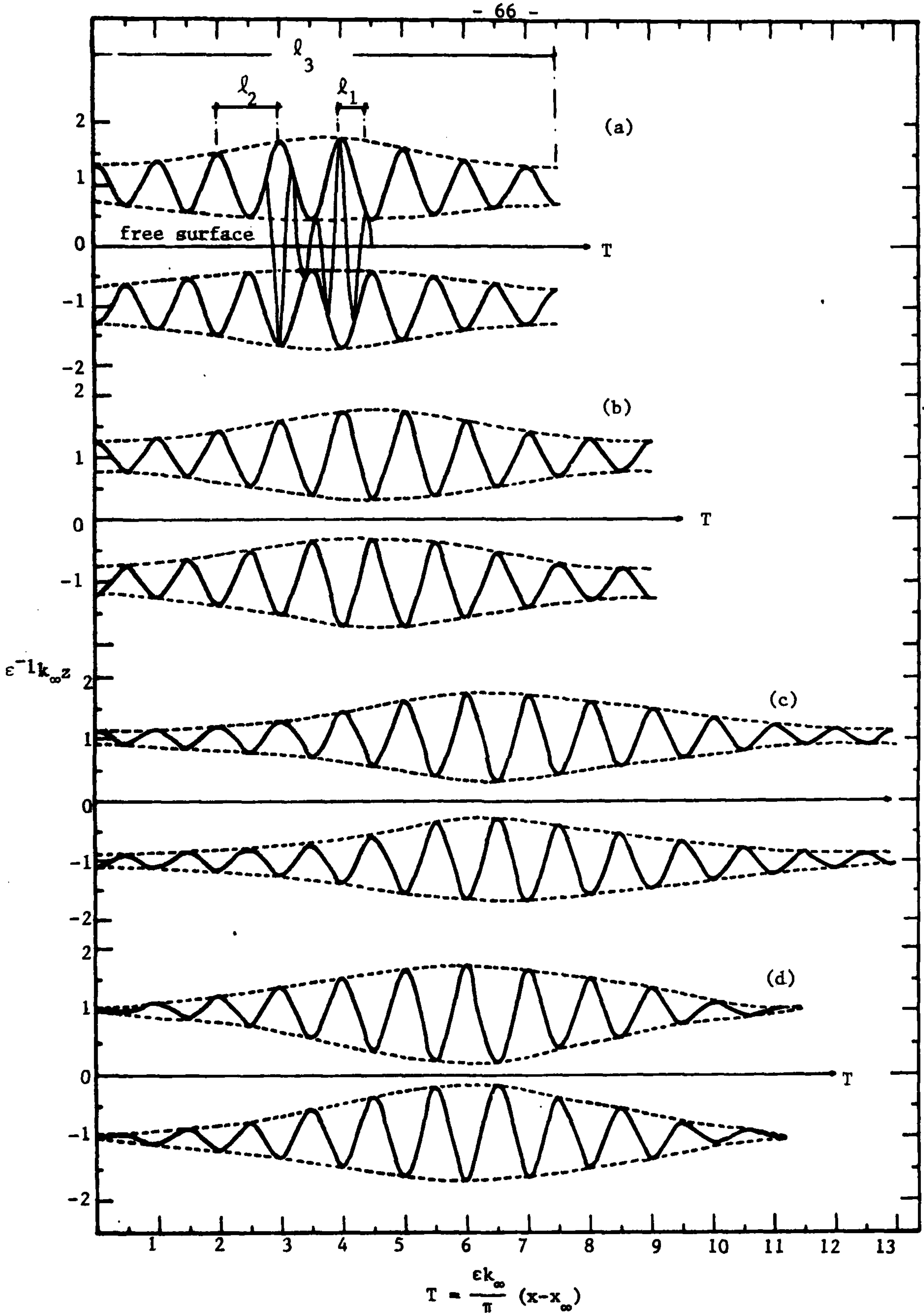


Figure 7

The group envelope (----) and wave envelope (——) at t=constant for P_o=2, I_o=0.1, ε= 0.2 at $k_\infty h$=(a)∞, (b) 11.2, (c) 5.7 and(d) 4.2.

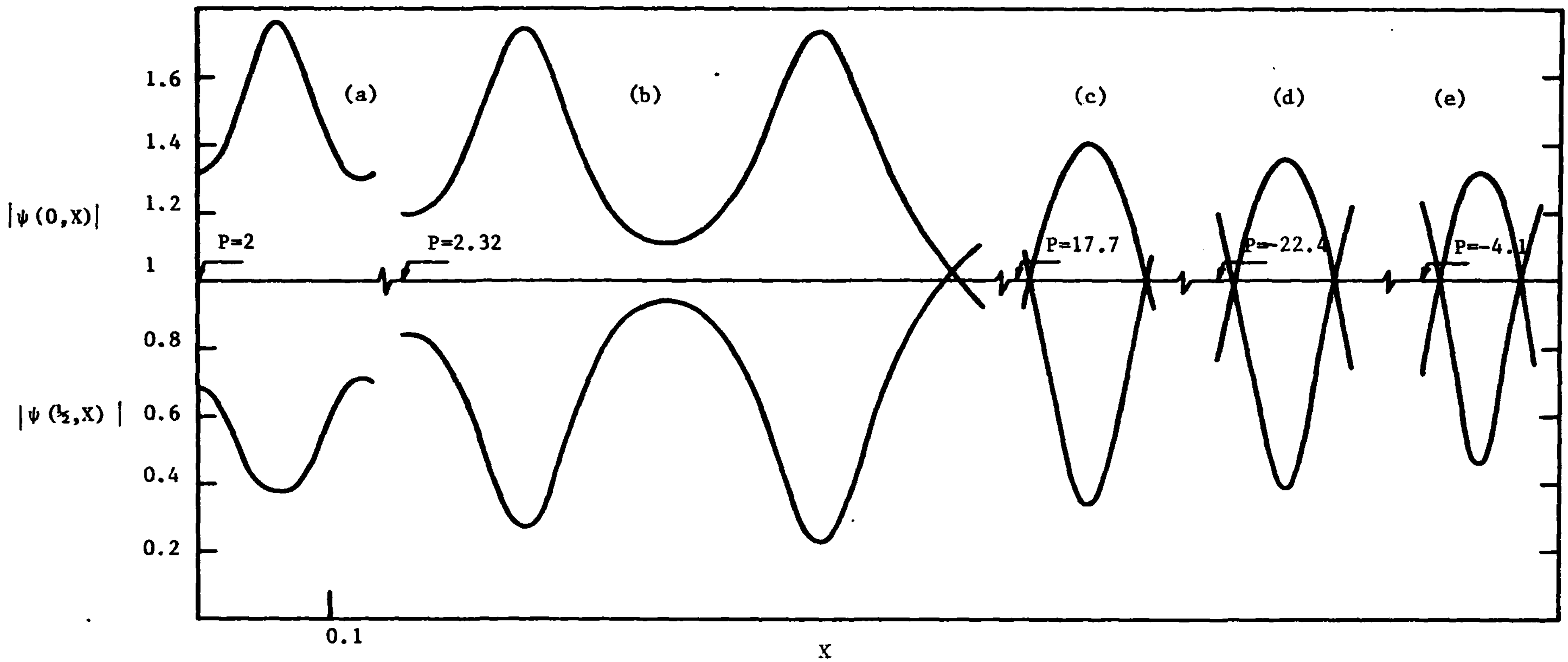


Figure 8

The group-envelopes for $P_o=2$, $I_o=0.1$, at $k_\infty h \approx$ (a) ∞, (b) 7.11-4.8, (C) 1.32-1.29, (d) 1.12-1.11, (e) 0.96-0.95.

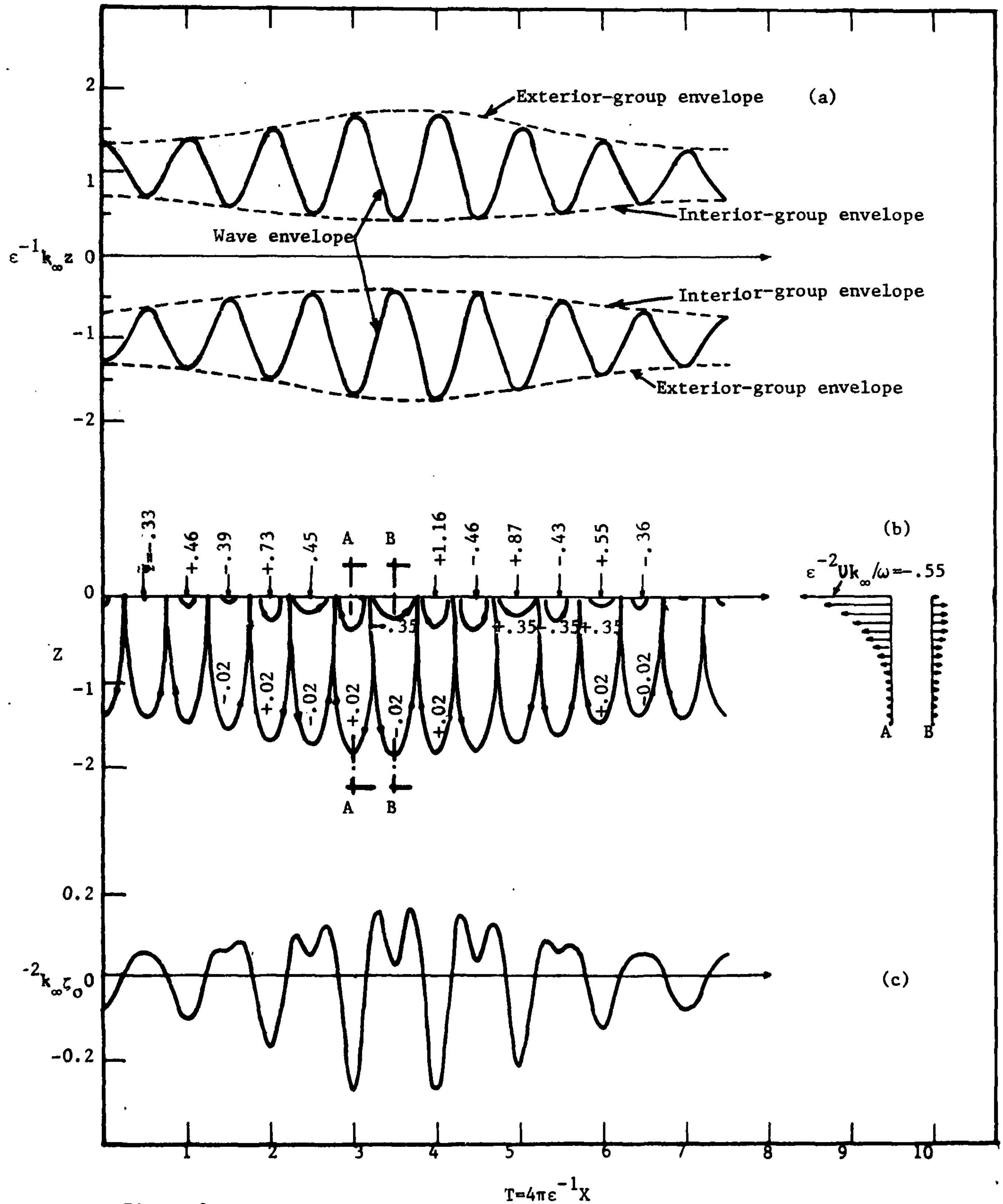
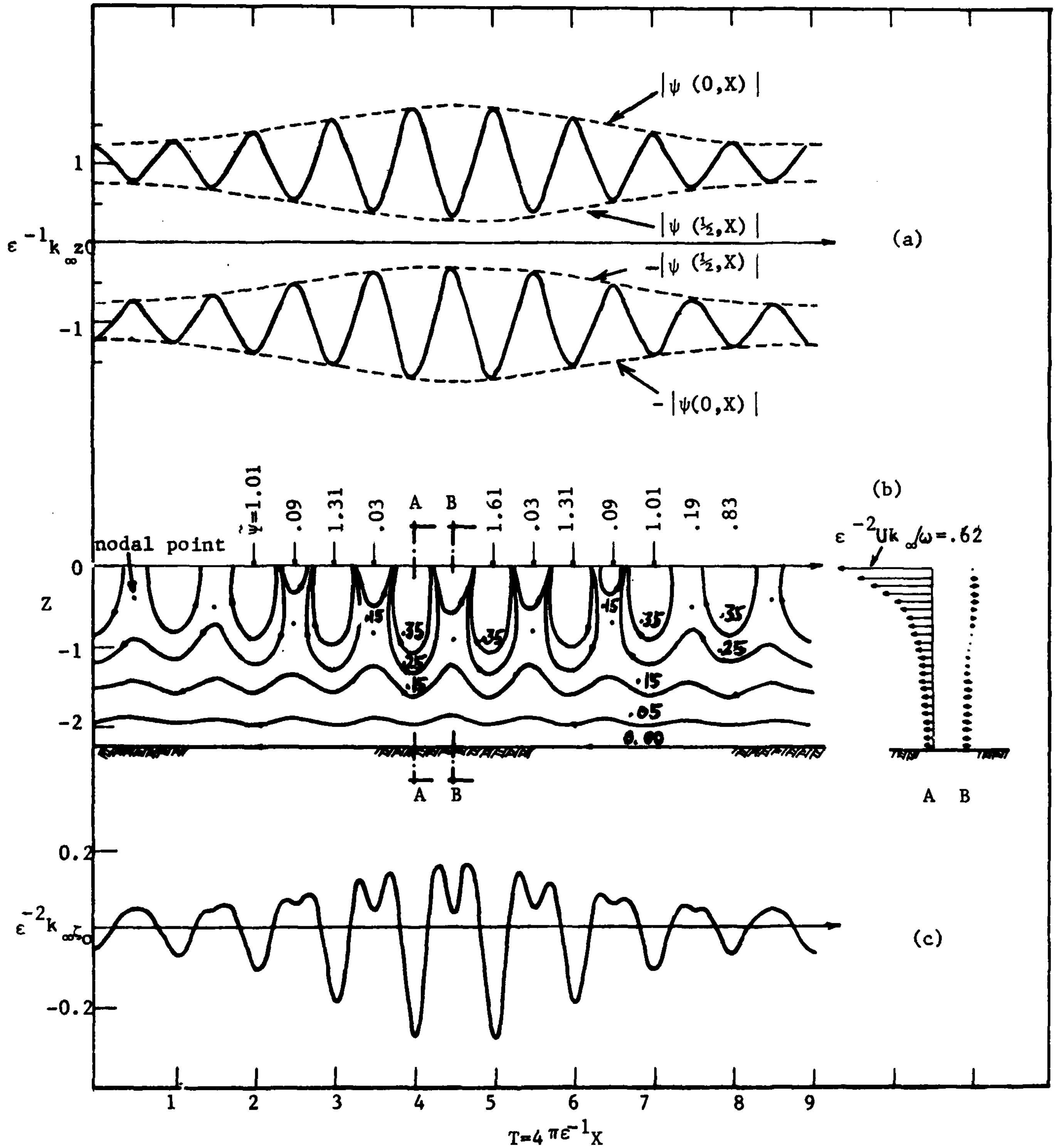


Figure 9

The flow field for P_o=2, I_o=0.1, ε =0.2 at $k_\infty h = \infty$ (a) group envelopes and wave envelope (b) mean flow stream-lines, (c) mean free surface.



<u>**Figure 10**</u>

As in Figure 9 for $k_\infty h = 11.2$

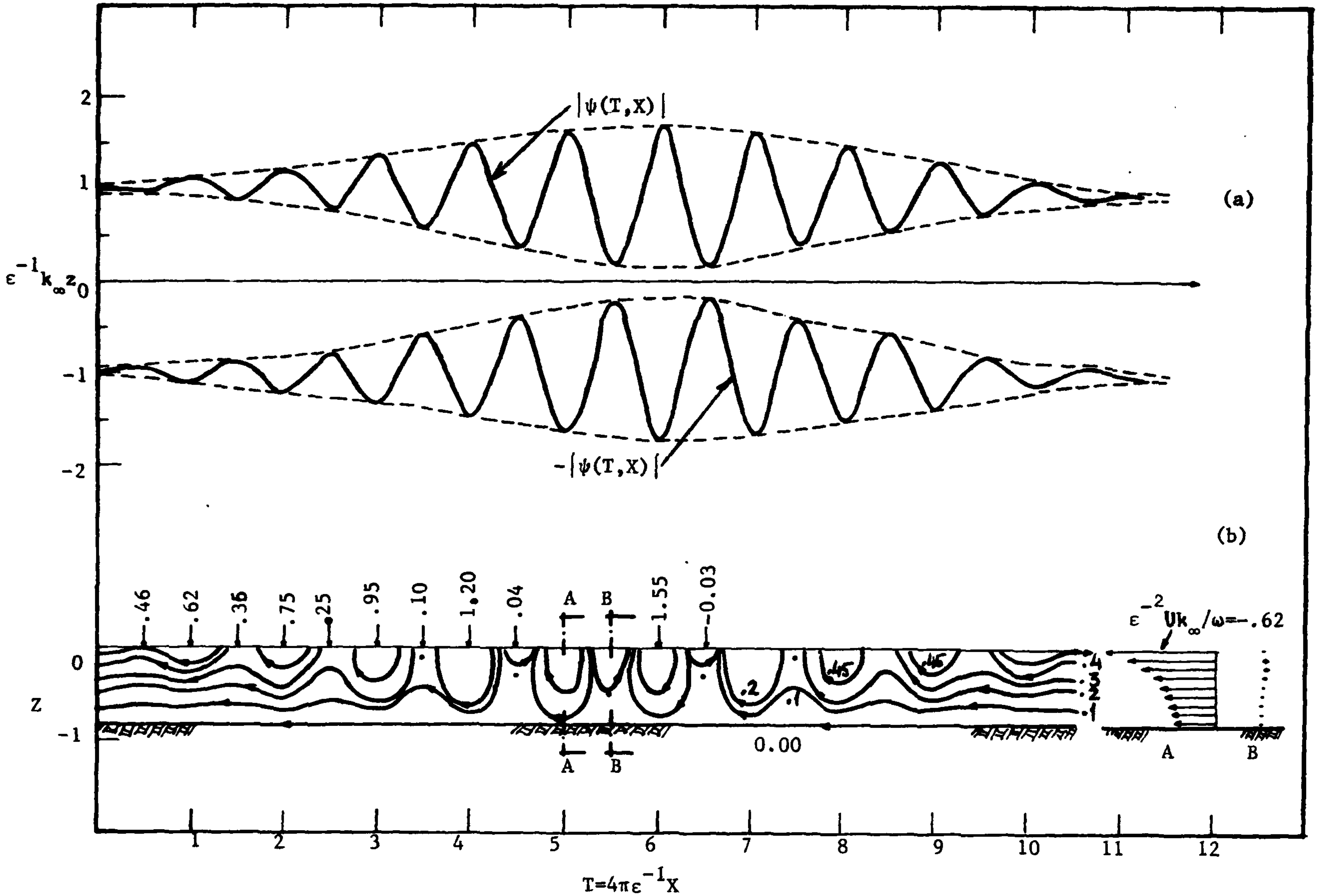
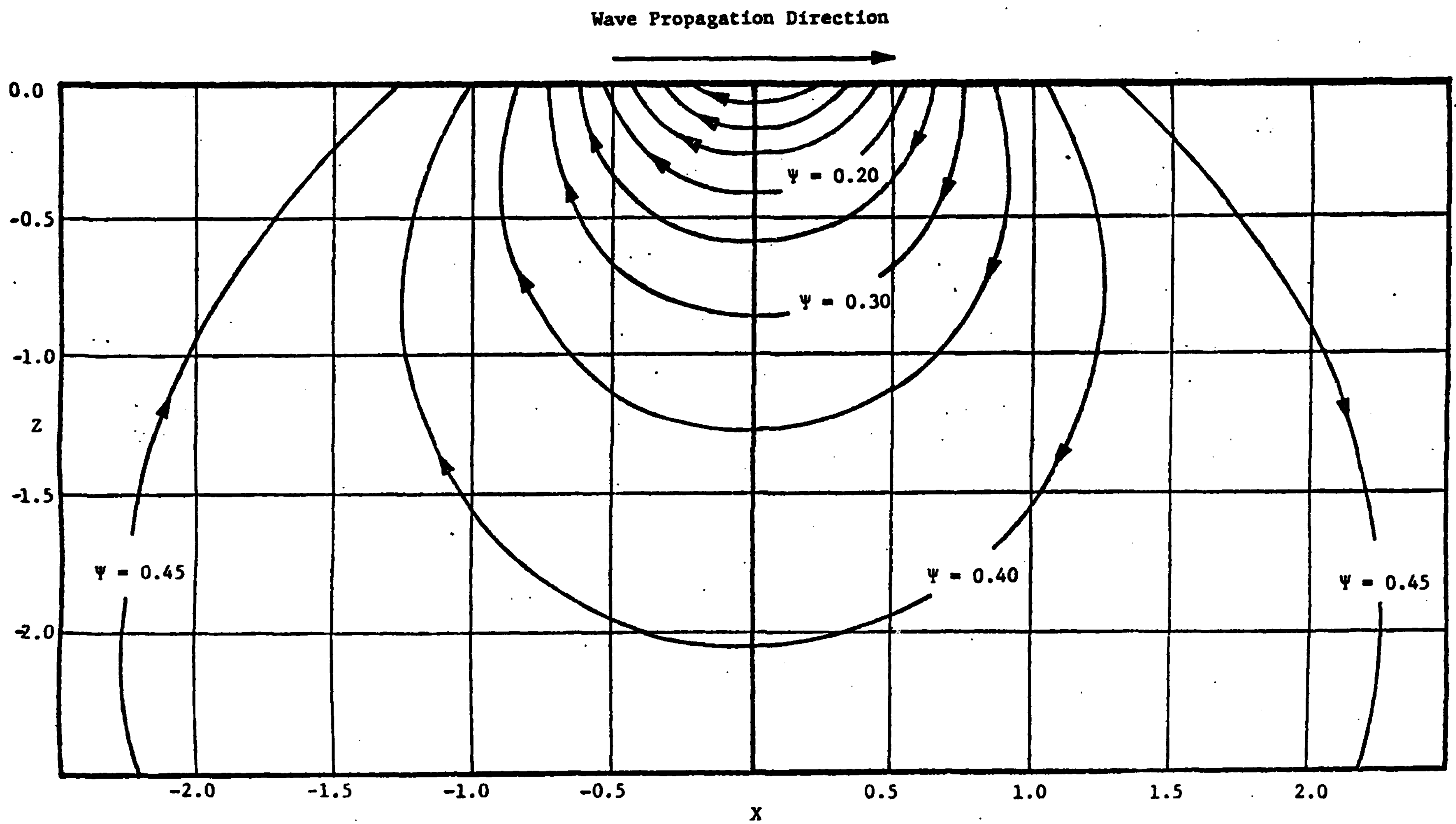


Figure 11

As in figure 9 for $k_\infty h = 4.2$



<u>Figure 12</u>

The wave-induced mean flow field, under a deep water envelope soliton

END

FILMED

10-85

DTIC